Analysis of Sliding Mode Controllers in the Frequency Domain

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Outline

1. Introduction
   - Describing Function Method
   - Oscillations in SMC systems

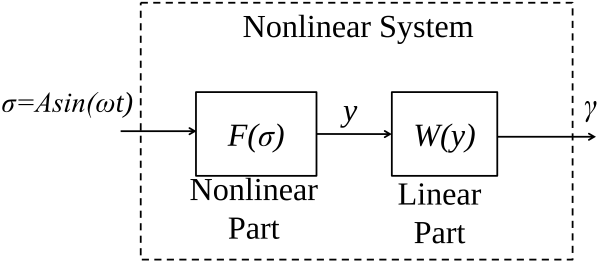
2. DF analysis of conventional SMC and 2-SMC algorithms
   - Conventional SMC: DF Analysis
   - Twisting Algorithm: DF Analysis
   - Terminal Control Law: DF Analysis
   - Super-twisting Algorithm: DF Analysis

3. Tolerance limits and Practical Stability Margins

Introduction DF

DF technique is:

- Applied to nonlinear systems where the nonlinear part can be separated from the linear part.
- Based on the hypothesis of law pass filter. i.e. that the input of the nonlinear part is sinusoidal.



Introduction DF

- Based on the Fourier series representation of the nonlinearity.

$$y = F(A\sin\omega t) = \frac{a_0}{2} + \Sigma_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

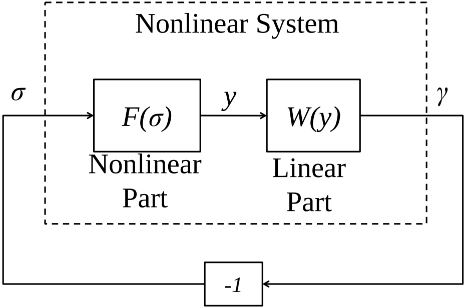
$$\begin{aligned} a_0 &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(A\sin\omega t)dt; \\ a_n &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(A\sin\omega t)\cos n\omega t dt; \\ b_n &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} F(A\sin\omega t)\sin n\omega t dt. \end{aligned}$$

Introduction DF

Main hypothesis: Linear part is a low pass filter.

Closed loop system, the output can be approximated

$$y = F(A \sin \omega t) \approx \frac{a_0}{2} + \Sigma_{n=1}^{\infty}(a_1 \cos \omega t + b_1 \sin \omega t)$$
$$a_2, a_3, \ldots \approx 0 \quad ; \quad b_2, b_3, \ldots \approx 0$$



Introduction DF

DF is an equivalent complex gain of the nonlinear part

$$F(\sigma) = N(A, \omega)\sigma$$

For symmetric nonlinearities

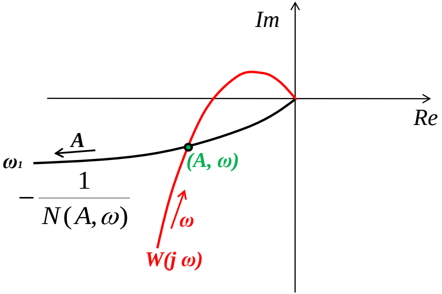
$$N(A,\omega) = \frac{\omega}{\pi A}\int_0^{2\pi/\omega} F(A\sin\omega t)\sin\omega t dt + j\frac{\omega}{\pi A}\int_0^{2\pi/\omega} F(A\sin\omega t)\cos\omega t dt$$

DF and Harmonic Balance

Harmonic Balance condition

$$1 + W(j\omega)N(A,\omega) = 0; \qquad W(j\omega) = -\frac{1}{N(A,\omega)}$$

- Identify oscillations
- Find frequency ω and amplitude A of the oscillations



Oscillations in SMC systems

NO ONE MODEL TAKES INTO ACCOUNT ALL SYSTEM DYNAMICS!!!
The phenomenon of **chattering** is caused by the inevitable existence of un-modeled dynamics.

The principal dynamics are the dynamics of the plant is a model that are used for controller design.

The un-modeled dynamics are not accounted during the SMC design; delays, actuators, sensors, etc.

The relative degree increases and the real sliding mode emerges, where the sliding variable contains a limit cycle (chattering) with finite frequency and finite amplitude.

DF in SMC

Systems driven by SMC analyzed can be the frequency domain, when the un-modeled dynamics are taken into account.

DF-HB technique is applied to identify limit cycles (chattering) and estimate their parameters, amplitude and frequency.

$$N(A,\omega) = \frac{\omega}{\pi A}\int_0^{2\pi/\omega} u(t)\sin\omega t dt + j\frac{\omega}{\pi A}\int_0^{2\pi/\omega} u(t)\cos\omega t dt$$

$N(A,\omega)$ is the DF of SMC algorithm.

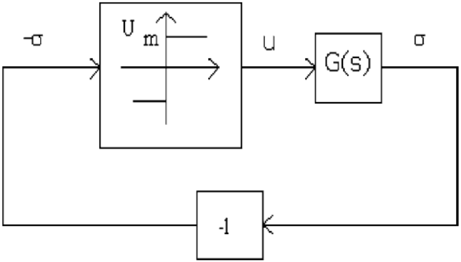


Figure: Block diagram of a linear system with relay control and ideal sliding

Replace the Laplace variable s by $j\omega$,

$$-\sigma = A_c \sin \omega_c t, \quad (1)$$

ω_c is frequency

Conventional SMC: DF Analysis

Amplitude A_c and Frequency ω_c have to satisfy the Harmonic Balance (HB) eq.

$$G(j\omega) = -\frac{1}{N(A,\omega)}. \tag{2}$$

For conventional SMC $N(A,\omega)$ DOES NOT DEPEND ON ω

$$N(A) = \frac{4U_m}{\pi A} \tag{3}$$

Example of Ideal SMC

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + u \\ \sigma &= x_1 + x_2 \end{aligned} \tag{4}$$

with control

$$u = -\text{sign}(\sigma) \tag{5}$$

Transfer function

$$G(s) = \frac{s+1}{s^2+s+1} \tag{6}$$

HB eq.

$$\mathrm{Re}\left[G(j\omega\right] = -\frac{\pi A}{4U_m} \tag{7}$$

Conventional SMC: DF analysis

Real part

$$\frac{1-\omega+\omega^2}{\left(1-\omega^2\right)^2+\omega^2} = -\frac{\pi A}{4U_m}$$

Imaginary part

$$\frac{\omega^2}{\left(1-\omega^2\right)^2+\omega^2} = 0 \tag{8}$$

Solution

$$A_c = 0,\ \omega_c \to \infty \tag{9}$$

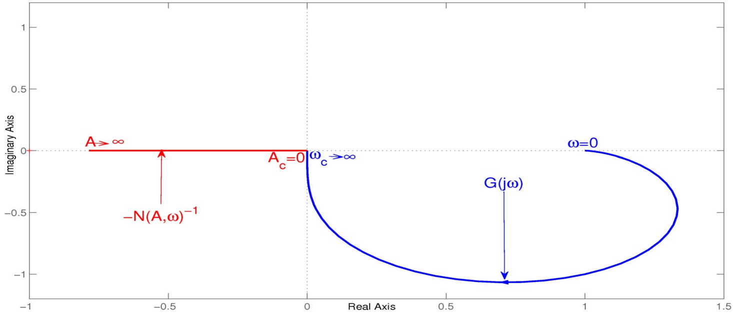


Figure: Graphical solution of the harmonic balance equation for system $G(s)$

Phase deficit is 90 grade. Finite time convergence!

Conventional SMC: DF analysis

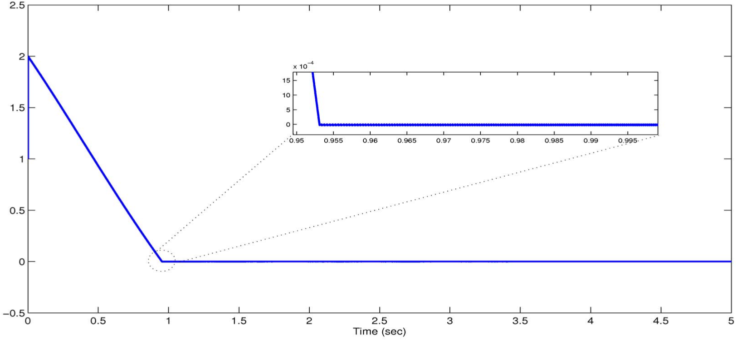


Figure: Surface

Phase deficit is 90 grade. Finite time convergence!

Conventional SMC: DF analysis

Real SMC

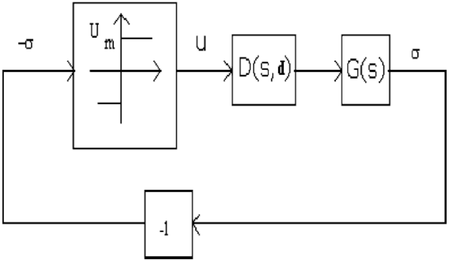


Figure: Block diagram of a linear system with Real SMC

$$D(j\omega, \mathbf{d})G(j\omega) = -\frac{1}{N(A,\omega)}, \quad N(A,\omega) = \frac{4U_m}{\pi A} \tag{10}$$

$D(j\omega, \mathbf{d})$ un-modelled dynamics

Example Real SMC

Plant

$$\dot{x}_1 = x_2;$$
$$\dot{x}_2 = -x_1 - x_2 + u;$$

Actuator

$$0.01\dot{u}_a = -u_a + u;$$
$$\sigma = x_1 + x_2;$$

Controller

$$u = -\text{sign}(\sigma) \tag{11}$$

Transfer function

$$D(s,d)G(s) = \frac{s+1}{(0.01s+1)(s^2+s+1)} \tag{12}$$

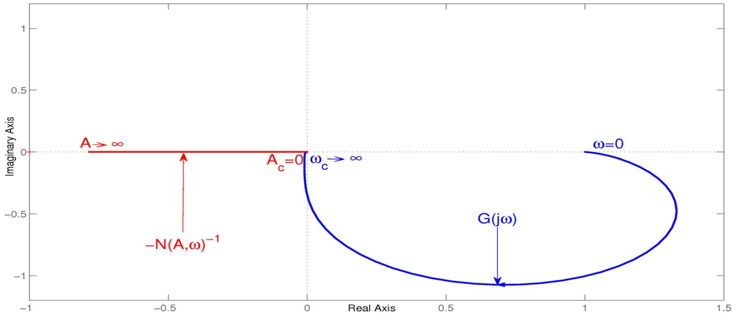


Figure: Graphical solution of the HB eq for system $D(s,d)G(s)$ plus 1st order actuator

The phase dificit is 0. Only asymptotic converence.

Conventional SMC: DF analysis

The phase dificit is 0. Only asymptotic converence.

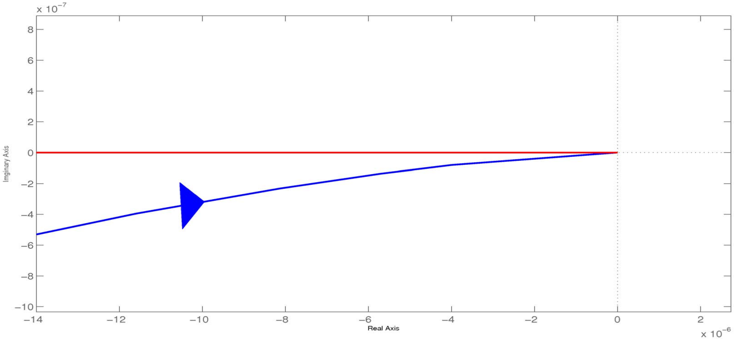


Figure: Zoom

Conventional SMC: DF analysis

The phase dificit is 0. Only asymptotic converence.

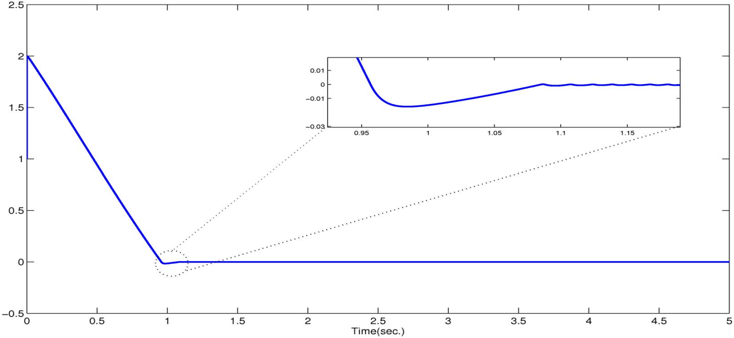
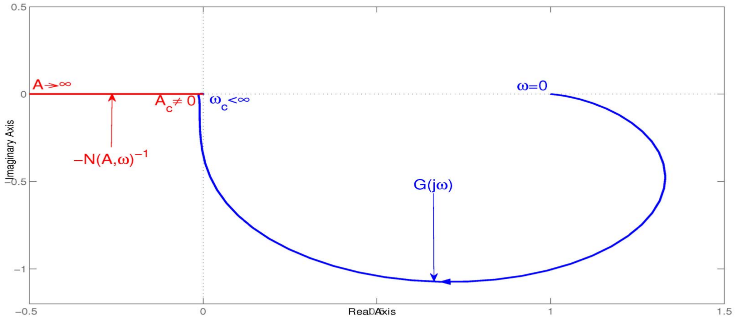


Figure: Surface

Conventional SMC: DF analysis

Plant plus 2nd order actuator

$$\dot{x}_1 = x_2 \quad ; \quad 0.0001\ddot{u}_a = -0.01\dot{u}_a - u_a + u$$
$$\dot{x}_2 = -x_1 - x_2 + u \quad ; \quad \sigma = x_1 + x_2$$
$$u = -\text{sign}(\sigma)$$



Conventional SMC: DF analysis

Plant plus 2nd order actuator

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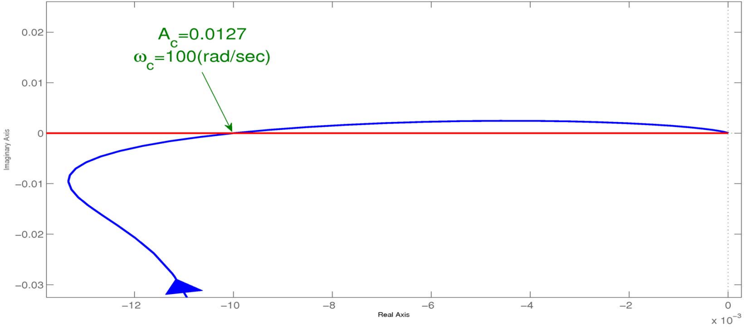

Conventional SMC: DF analysis

Plant plus 2nd order actuator

$$\dot{x}_1 = x_2 \quad ; \quad 0.0001\ddot{u}_a = -0.01\dot{u}_a - u_a + u$$
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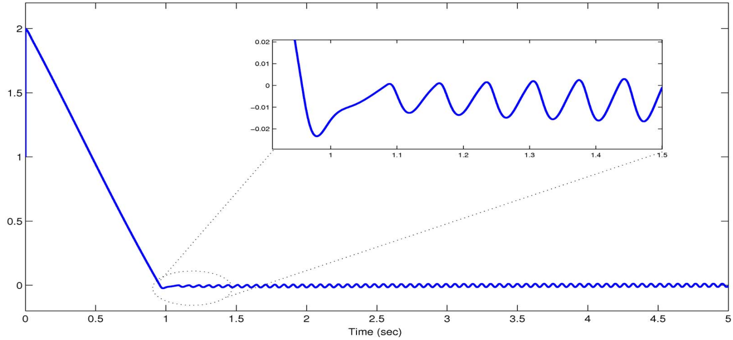


Twisting and its DF

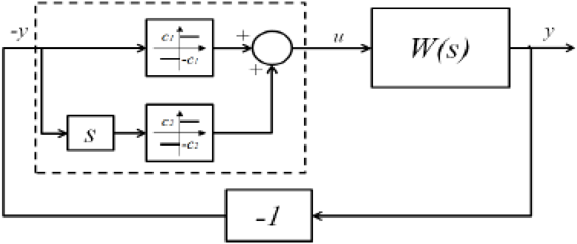
Twisting Algorithm

$$\ddot{x} = u;$$
$$u = -c_1 sign(x) - c_2 sign(\dot{x}),$$

with $c_1 > c_2 > 0$.

DF

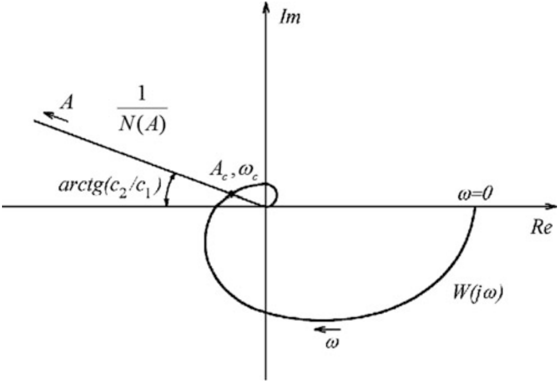
$$N(A) = N_1 + sN_2 = \frac{4}{\pi A}(c_1 + jc_2),$$



Twisting and its DF

HB eq.

$$W(j\omega) = \pi A \frac{-c_1 + jc_2}{4(c_1^2 + c_2^2)},$$



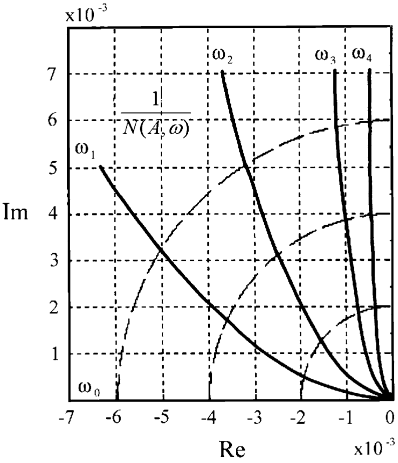
The phase dificit is $arctg(c_2/c_1)$. Finite-time convergence

Terminal Control Law and its DF

Terminal Control Law

$$\ddot{x} = u;$$
$$u = -\alpha sign(\dot{x} + \beta |x|^{\rho} sign(x)),$$

with $0.5 < \rho < 1$.



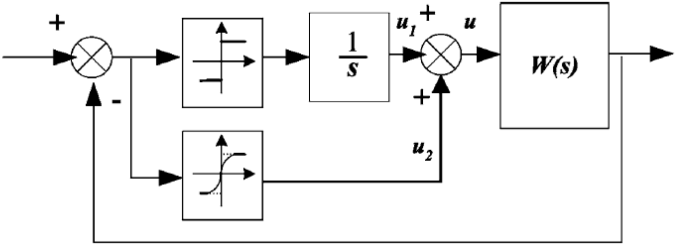
Super-twisting and its DF

ST algorithm

$$\begin{aligned} \dot{x} &= u; \\ u &= -\beta|\sigma|^{1/2} sign(\sigma) + u_s, \\ \dot{u}_s &= -\alpha sign(\sigma), \end{aligned}$$

DF

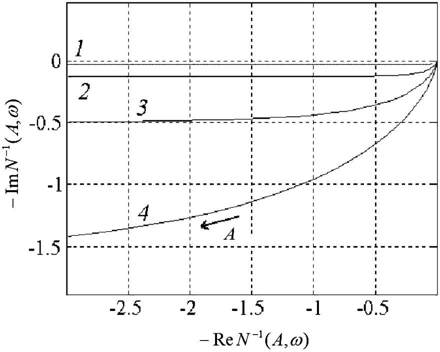
$$N(A,\omega) = \frac{4\alpha}{\pi A j\omega} + \frac{1.1128\beta}{\sqrt{A}},$$



Super-twisting and its DF

HB eq.

$$W(j\omega) = -\frac{0.8986\frac{\sqrt{A}}{\beta} + j1.1329\frac{\alpha}{\beta^2\omega}}{1 + 1.3092\frac{\alpha^2}{\beta^2 A\omega^2}}$$



Existence of the periodic solutions

Write HB eq. as: $N(A) = -W^{-1}(j\omega)$,

$$\frac{4\gamma}{\pi A}\frac{1}{j\omega} + 1.1128\frac{\lambda}{\sqrt{A}} = -W^{-1}(j\omega). \tag{13}$$

Consider the real part of both sides

$$1.1128\frac{\lambda}{\sqrt{A}} = -\operatorname{Re} W^{-1}(j\omega) \tag{14}$$

Eliminating A from eqs. (13)-(14),

$$\Psi(\omega) = \frac{4\gamma}{\pi\omega}\frac{1}{\operatorname{Im} W^{-1}(j\omega)} - \left(\frac{1.1128\lambda}{\operatorname{Re} W^{-1}(j\omega)}\right)^2 = 0. \tag{15}$$

Eq. (15) has ONLY one unknown variable, ω.

Existence of the periodic solutions

Once ω is obtained from Eq. (15) amplitude, A_c can be computed as:

$$A_c = \frac{4\gamma}{\pi\omega_c}\frac{1}{\mathrm{Im}\, W^{-1}(j\omega_c)}. \tag{16}$$

Super-twisting and its DF

Stability of periodic solution

If the following inequality holds then the periodic solution given by Equation (15) is locally stable:

$$\mathrm{Re}\, \frac{h_1(A,\omega)}{h_2(A,\omega) + N(A,\omega)\frac{\partial \ln W(s)}{\partial s}\ |_{s=j\omega}} < 0, \tag{17}$$

where $h_1(A,\omega) = \frac{1.1128\lambda}{2A^{\frac{3}{2}}} - j\frac{4\gamma}{\pi\omega A^2}$, $h_2(A,\omega) = \frac{4\gamma}{\pi\omega^2 A}$

Stability of periodic solution

Proof:

- Assume that the HB eq. holds for small perturbations.
- Damped oscillation of the complex frequency $j\omega + (\triangle\sigma + j\Delta\omega)$ corresponds to the modified amplitude $(A + \triangle A)$:

$$N(A + \triangle A, j\omega + (\triangle\sigma + j\Delta\omega))W(j\omega + (\triangle\sigma + j\Delta\omega)) = -1. \tag{18}$$

  $N(A, \omega)$ is DF of Super-twisting.
- Find the conditions when $\Lambda = \triangle\sigma / \triangle A$ is negative.

Stability of periodic solution

Proof(continue):

- Take the derivative of (18) with respect to $\triangle A$ and write an equation for the amplitude perturbation $\triangle A$.

$$\left\{ \frac{dN(\triangle A, \triangle\sigma, \Delta\omega)}{d\triangle A}|_{\triangle A=0} W(j\omega) + \frac{dW(\triangle\sigma, \Delta\omega)}{d\triangle A}|_{\triangle A=0} N(A,\omega) \right\} \triangle A = 0. \quad (19)$$

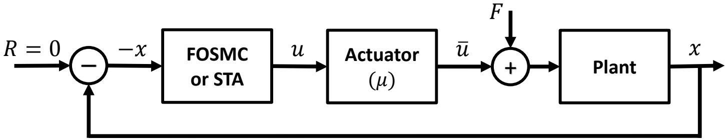
- Take derivatives of N and W, and consider them composite functions:

$$\frac{dN(\triangle A, \triangle\sigma, \Delta\omega)}{d\triangle A}|_{\triangle A=0} = -j\frac{4\gamma\omega}{\pi A^2} - \frac{1.1128\lambda}{2A^{\frac{3}{2}}} + \frac{4\gamma A}{\pi\omega^2}\left(\frac{d\triangle\sigma}{d\triangle A} + j\frac{d\triangle\omega}{d\triangle A}\right). \quad (20)$$

$$\frac{dW}{d\triangle A}|_{\triangle A=0} = \frac{dW}{ds}|_{s=j\omega}\left(\frac{d\triangle\sigma}{d\triangle A} + j\frac{d\triangle\omega}{d\triangle A}\right) \quad (21)$$

- Solve eq. (19) for$\left(\frac{d\triangle\sigma}{d\triangle A} + j\frac{d\triangle\omega}{d\triangle A}\right)$ and taking account of (20) and (21), an analytical formula is obtained, where the real part is (17).

Is It Reasonable to Substitute Discontinuous SMC by Continuous HOSMC?



Plant

$$\dot{x}(t) = \bar{u}(t) + F(t)$$

Actuator

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\mu^2} & -\frac{2}{\mu} \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \frac{1}{\mu^2} \end{bmatrix} u(t)$$

$$\bar{u}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} z(t)$$

Assumption 1

The parasitic dynamics (Actuator) is not required for the design of the SMC/HOSMC gains and its effects can be measured by the ATC μ.

Considered SMC Algorithms and Disturbances

FOSMC

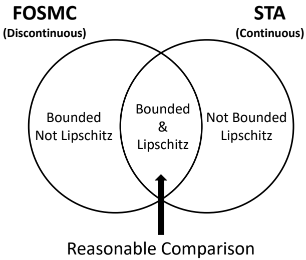
$$u = -M\,\mathrm{sign}(x)$$

where $M = 1.1\delta$.

STA [Levant (1993)]

$$\begin{aligned} u &= -k_1|x|^{1/2}\,\mathrm{sign}(x) + v \\ \dot{v} &= -k_2\,\mathrm{sign}(x) \end{aligned}$$

where $k_1 = 1.5\sqrt{L},\ k_2 = 1.1L$.



Disturbance form

$$F = \alpha\sin(\Omega t)$$

where the upperbounds

$$\begin{aligned} |F| &\leq \delta = \alpha \\ |\dot{F}| &\leq \Delta = \alpha\,\Omega \end{aligned}$$

are assumed known.

Simulations Results for Some Values of ATC and Increasing Ω

Control \ Ω		**1**	**10**	**100**
Discontinuous Control				
FOSMC	$\mu = 10^{-1}$	1.366×10^{-1}	1.692×10^{-1}	0.934×10^{-1}
	$\mu = 10^{-2}$	1.092×10^{-2}	1.361×10^{-2}	1.692×10^{-2}
	$\mu = 10^{-3}$	1.064×10^{-3}	1.096×10^{-3}	1.362×10^{-3}
Continuous Control				
STA	$\mu = 10^{-1}$	1.243×10^{-1}	8.663×10^{-1}	6.4041
	$\mu = 10^{-2}$	9.431×10^{-4}	1.302×10^{-2}	8.694×10^{-2}
	$\mu = 10^{-3}$	8.915×10^{-6}	9.445×10^{-5}	1.343×10^{-3}

Table: Sliding-Mode Amplitude Accuracy

Discussion Aspects

Professor V. Utkin Hypothesis

Simulations confirms that for any value of ATC there exist a bounded disturbance for which the amplitude of possible oscillations produced by FOSMC is lower than the obtained applying STA.

Hypothesis 2

It should exists a value of ATC for which the amplitude of chattering produced by FOSMC and STA are the same.

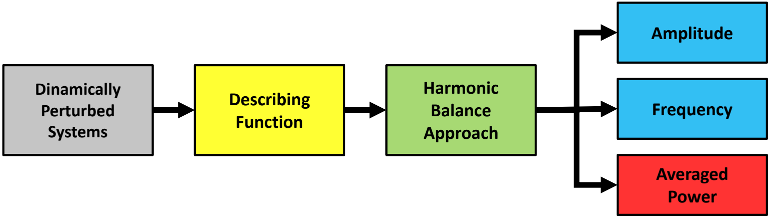
Hypothesis 3

For any bounded and Lipschitz disturbance, the amplitude of possible oscillations produced by STA may be less than the obtained using FOSMC if the actuator dynamics is fast enough.

Methodology

The parameters that characterizes the chattering of the steady-state behavior of the nominal system ($F = 0$) are:

1. Amplitude of periodic motion (A)
2. Frequency of periodic motion (ω)
3. Average power (P)



DF Analysis

Assumption 2 (Low pass filter hypothesis)

The dynamically perturbed system (Actuator-Plant) $W(s)$ has *low pass filter* characteristics with respect to the higher harmonics of the output x. Hence the output of the system converges to a periodic motion [Gelb (1968)], [Boiko (2009)], which can be well-approximated by its first-harmonic,

$$\begin{aligned} x &= A\sin(\omega t), \\ \dot{x} &= A\omega\cos(\omega t). \end{aligned}$$

Harmonic Balance Approach

Parameters of a possible periodic motion, amplitude A and frequency ω, can be found by solving the Harmonic Balance equation (see for example [Gelb (1968)], [Atherton (1975)])

$$N(A,\omega)W(j\omega) = -1$$

where $N(A,\omega)$ is the DF of the non-linearity (FOSMC or STA).

Average Power Approach

L_p-chattering [Levant (2010)]

$$\text{chatt}_{L_p}(x) = \left(\int_0^T \dot{x}^p(\tau) d\tau \right)^{1/p}$$

Drawbacks

- There is no chattering in ideal sliding-mode motion!
- How to compute chatt_{L_p}?

$$\Downarrow$$

A novel approach: Average Power

$$\mathrm{P} = \frac{1}{T} \int_0^T \dot{x}^2(\tau) d\tau = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \Big(A\omega \cos(\omega\tau) \Big)^2 d\tau = \frac{A^2\omega^2}{2}$$

Preliminaries

Let the dynamically perturbed system (actuator-plant)

$$W(s) = \frac{1}{s(\mu s + 1)^2}$$

FOSMC

$$u = -M \operatorname{sign}(x)$$

STA [Levant (1993)]

$$\begin{aligned} u &= -k_1 |x|^{1/2} \operatorname{sign}(x) + v \\ \dot{v} &= -k_2 \operatorname{sign}(x) \end{aligned}$$

DF

$$N(A) = \frac{4M}{\pi A}$$

Note: The DF of FOSMC does not depend on frequency ω.

DF [Boiko (2009)]

$$N(A, \omega) = \frac{1.1128 k_1}{A^{1/2}} - j \frac{4 k_2}{\pi A \omega}$$

Chattering Parameters Estimated by HB

FOSMC [?]

$$A = \mu\left(\frac{2M}{\pi}\right)$$

$$\omega = \frac{1}{\mu}$$

$$P = \frac{2M^2}{\pi^2}$$

Note: The Average Power produced by FOSMC does not depend on ATC μ.

STA [?]

$$A = \mu^2\left(\frac{1}{2}\cdot\frac{(1.1128k_1)^2+\frac{16}{\pi}k_2}{1.1128k_1}\right)^2$$

$$\omega = \frac{1}{\mu}\left(\frac{(1.1128k_1)^2}{(1.1128k_1)^2+\frac{16}{\pi}k_2}\right)^{1/2}$$

$$P = \frac{\mu^2}{4}\left(\frac{1}{2}\cdot\frac{(1.1128k_1)^2+\frac{16}{\pi}k_2}{(1.1128k_1)^{2/3}}\right)^3$$

Minimum Amplitude for each $k_2 > \Delta$

$$\bar{k}_1 = \left(\frac{16k_2}{\pi(1.1128)^2}\right)^{1/2} = 2.028\sqrt{k_2}$$

Proposed STA Gains†

$$k_1 = 2.127\sqrt{\Delta}$$
$$k_2 = 1.1\Delta$$

† Sufficient stability conditions [?] are satisfied:

$$k_1 > 1.449\sqrt{\Delta}$$
$$k_2 = 1.1\Delta$$

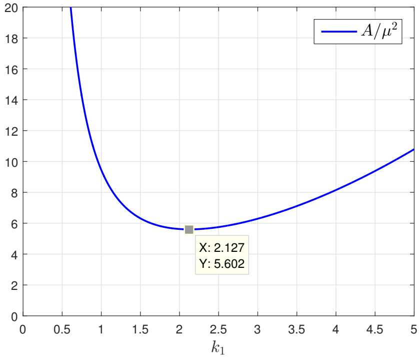


Figure: Amplitude as Function of k_1

Selection of STA Gains Minimize the Average Power

Minimum AP for a given $k_2 > \Delta$

$$\bar{k}_1 = \left(\frac{8k_2}{\pi(1.1128)^2}\right)^{1/2} = 1.434\sqrt{k_2}$$

Proposed STA Gains†

$$k_1 = 1.504\sqrt{\Delta}$$
$$k_2 = 1.1\Delta$$

† Sufficient stability conditions [?] are satisfied:

$$k_1 > 1.449\sqrt{\Delta}$$
$$k_2 = 1.1\Delta$$

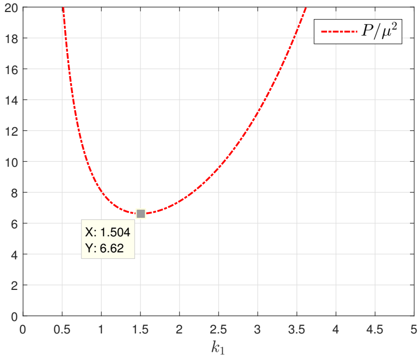


Figure: Average Power as Function of k_1

Chattering Parameters as Function of μ

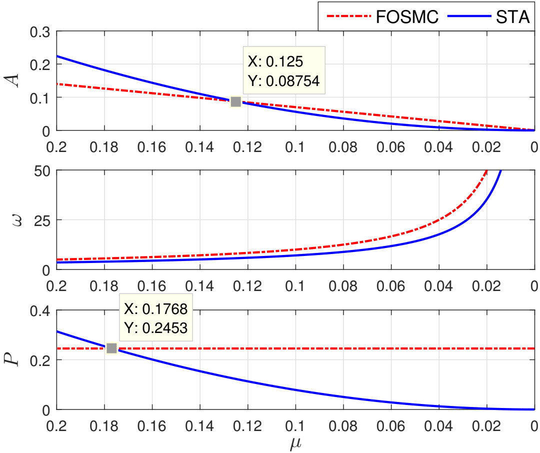


Figure: Chattering Parameters as Function of ATC μ

Amplitude Discussion

Result 1

There exist a value of ATC for which the amplitude of possible oscillations are the same according with HB approach,

$$\mu^* = \frac{8M(1.1128k_1)^2}{\pi\left((1.1128k_1)^2 + \frac{16}{\pi}k_2\right)^2}$$

Frequency Discussion

Result 2

The frequency of possible oscillations is always lower for the STA than the obtained using FOSMC.

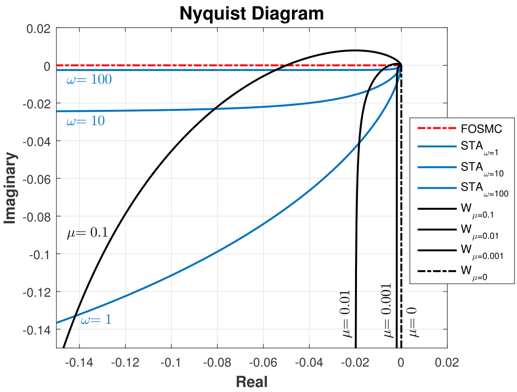


Figure: Graphical Solution of HB equation

Result 3

There exist a value of ATC for which the average power are the same according with HB approach,

$$\mu^{\star} = \frac{8M(1.1128k_1)}{\pi\left((1.1128k_1)^2 + \frac{16}{\pi}k_2\right)^{3/2}}$$

Comparison Example

Let a matched disturbance F,

$$F = \alpha \sin(\Omega t) \quad \Rightarrow \quad \begin{cases} |F| \leq \delta = \alpha \\ |\dot{F}| \leq \Delta = \alpha\,\Omega \end{cases}$$

Same Amplitude Order

$$\mu^* = \frac{8M(1.1128k_1)^2}{\pi\left((1.1128k_1)^2 + \frac{16}{\pi}k_2\right)^2} = 0.125\frac{\delta}{\Delta} = 0.125\frac{1}{\Omega}$$

Considered ATC

$$\mu^* < \mu_1 = 0.25\frac{1}{\Omega}$$

Slow actuator

$$\mu^* > \mu_2 = 0.0833\frac{1}{\Omega}$$

Fast actuator

Simulations for some values of Ω

Control \ Ω		1	10	100
Discontinuous Control				
FOSMC	μ_1	1.6326	1.6224×10^{-1}	1.6226×10^{-2}
	μ^*	1.7644×10^{-1}	1.9018×10^{-2}	1.8969×10^{-3}
	μ_2	9.4217×10^{-2}	9.4311×10^{-3}	9.4872×10^{-4}
Continuous Control				
STA	μ_1	2.2492	2.6933×10^{-1}	2.7061×10^{-2}
	μ^*	1.3229×10^{-1}	1.3516×10^{-2}	1.3518×10^{-3}
	μ_2	4.8421×10^{-2}	4.8374×10^{-3}	4.8573×10^{-4}

Table: Sliding-Mode Output Accuracy Increasing the Disturbance Frequency Ω.

Discussion of Chattering Magnitude

Simulation results confirms:

- For any value of disturbance frequency Ω should be a critical value of ATC μ^* for which the magnitude of chattering is the same when FOSMC or STA are applied.
- If ATC is greater than μ^* (for example μ_1), then

$$A_{\text{FOSMC}} < A_{\text{STA}}$$

- If ATC is lower than μ^* (for example μ_2), then

$$A_{\text{FOSMC}} > A_{\text{STA}}$$

Professor V. Utkin Example

Consider the following case

$$\delta = \Delta = 60 \quad \Rightarrow \quad \begin{cases} M = 1.1\delta \\ k_1 = 2.127\sqrt{\Delta} \quad \text{and} \quad k_2 = 1.1\Delta \end{cases}$$

Chattering Parameters Estimated by HB

- **FOSMC**

$$A = 42.017\mu, \quad \omega = \frac{1}{\mu}, \quad P = 882.7102.$$

- **STA**

$$A = 336.135\mu^2, \quad \omega = \frac{1}{\mu\sqrt{2}}, \quad P = 28246.93\mu^2.$$

Same Amplitude

$$\mu^* = 0.125$$

Same Average Power

$$\mu^\star = 0.1768$$

Simulations for some values of μ

Control \ μ		0.2	0.1768	0.125	0.1
Discontinuous Control					
FOSMC	**A**	8.6899	7.6819	5.4312	4.3450
	$\boldsymbol{\omega}$	4.8900	5.5317	7.8240	9.7800
	P	926.899	926.899	926.899	926.899
Continuous Control					
STA	**A**	13.5615	10.5999	5.2987	3.3911
	$\boldsymbol{\omega}$	3.5153	3.9764	5.6242	7.0302
	P	1152.394	900.6406	450.360	288.2422

Table: Comparison of Chattering Parameters

Discussion about the Chattering Parameters Based on HB

HB allows to confirm:

- Given δ and Δ upperbounds of disturbance and time-derivative disturbance, it should be exist an ATC μ^* for which the amplitude of possible oscillations are the same. Also

$$\text{if} \quad \begin{array}{lcl} \mu > \mu^* & \Rightarrow & A_{\mathrm{FOSMC}} < A_{\mathrm{STA}} \,, \\ \mu < \mu^* & \Rightarrow & A_{\mathrm{FOSMC}} > A_{\mathrm{STA}} \,. \end{array}$$

- Given δ and Δ upperbounds of disturbance and time-derivative disturbance, it should be exist an ATC $\mu^\star$ for which the average power are the same. Also

$$\text{if} \quad \begin{array}{lcl} \mu > \mu^\star & \Rightarrow & P_{\mathrm{FOSMC}} < P_{\mathrm{STA}} \,, \\ \mu < \mu^\star & \Rightarrow & P_{\mathrm{FOSMC}} > P_{\mathrm{STA}} \,. \end{array}$$

Average Power: Passivity Approach

Average Power

$$\bar{p}(t) = \bar{u}(t)x(t) \quad \Rightarrow \quad \bar{P} = \frac{1}{T}\int_0^T p(\tau)d\tau = \frac{4A^2\omega}{\pi}$$

FOSMC

$$\bar{P} = \mu\left(\frac{16M^2}{\pi^3}\right)$$

STA

$$\bar{P} = \mu^3\left(\frac{\left[(1.1128k_1)^2 + \frac{16}{\pi}k_2\right]^{7/2}}{4\pi(1.1128k_1)^3}\right)$$

Selection of STA Gains to Minimize the Average Power

Minimum AP for $k_2 > \Delta$

$$\bar{k}_1 = \left(\frac{12k_2}{\pi(1.1128)^2}\right)^{1/2} = 1.7563\sqrt{k_2}$$

Proposed STA Gains†

$$k_1 = 1.842\sqrt{\Delta}$$
$$k_2 = 1.1\Delta$$

† Sufficient stability conditions [?] are satisfied:

$$k_1 > 1.449\sqrt{\Delta}$$
$$k_2 = 1.1\Delta$$

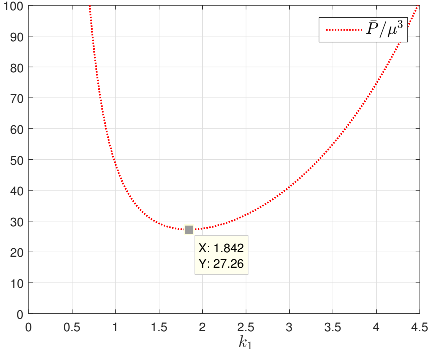
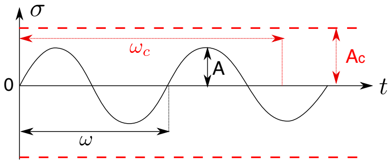


Figure: Average Power as Function of k_1

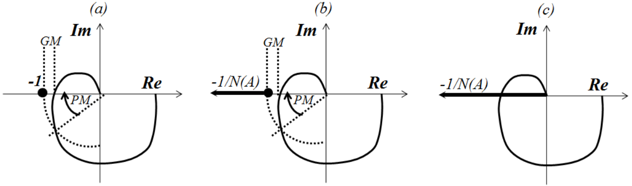
Tolerance limits

The frequency $0 < \omega_c < \infty$ and amplitude $A_c > 0$ are the Tolerance Limits (TL) of the acceptable limit cycle of the output σ, so that its self-sustained oscillations with the amplitudes $A \leq A_c$ and the frequencies $\omega \geq \omega_c$ yield the acceptable behavior of the closed loop system.

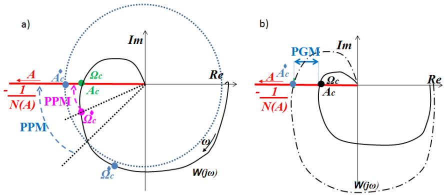


Practical Stability Margins

Classical stability margins can not be applied



Practical Phase Margin (PPM) and Practical Gain Margin (PGM)



PGM and PPM

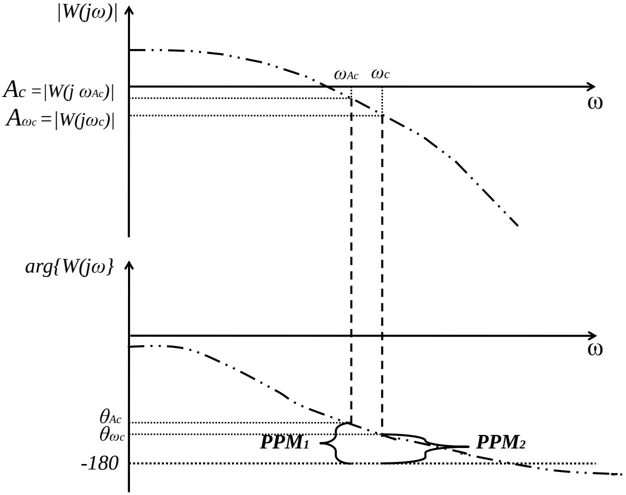
Performance Gain Margin (PGM)

The PGM in the closed loop system controlled by SMC is the maximum additional gain added to the frequency characteristic of the linear (linearized) plant $W(j\omega)$, while the sliding variable σ (which is the output of the closed loop system) exhibits a limit cycle with marginally reached amplitude $A = A_c$ and/or frequency $\omega = \omega_c$ whatever comes first.

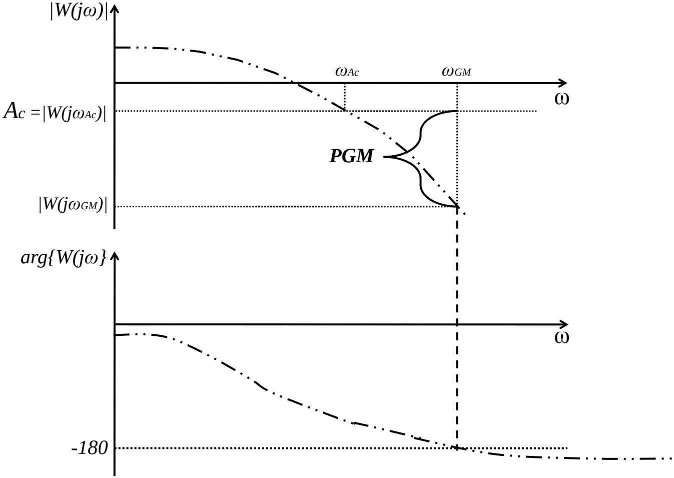
Performance Fase Margin (PPM)

The PPM in the closed loop system controlled by SMC is the maximal additional phase shift that can be added to the frequency characteristic of the linear (linearized) plant $W(j\omega)$, while the sliding variable σ (which is the output of the closed loop system) exhibits a limit cycle with marginally reached amplitude $A = A_c$ and/or frequency $\omega = \omega_c$ whatever comes first.

PPM via Bode Diagram



PGM via Bode Diagram



Compensator

$$W_c(s) = \frac{\tau s + 1}{\beta \tau s + 1},$$

β attenuation parameter. $W_c(s)$ is phase-lead $0 < \beta < 1$, and is phase-lag $\beta > 1$.

- Obtain the performance margins of the system controlled by SMC.
- Determine the maximum phase-lead angle of compensator as

$$\phi_m = PSPM_c{}^\circ - PSPM_{un}{}^\circ + \langle 5, 12 \rangle^\circ,$$

  where $PSPM_{un}$ is the PPM of the uncompensated system, $PSPM_c$ is the desired $PSPM$ and $\langle 5, 12 \rangle$ means an interval.
- Obtain the parameter β that satisfies equation

$$\sin \phi_m = \frac{1 - \beta}{1 + \beta}.$$

- Identify from the amplitude-frequency Bode plot of the uncompensated system the magnitude that is equal to

$$-\left[20\log\left(\frac{1}{\sqrt{\beta}}\right)+20\log\left(\left|-\frac{1}{N(A_c,\omega)}\right|\right)\right],$$

and it is associated with the frequency ω_m.

- Calculate the pole and zero of $W_c(j\omega)$ as

$$\text{Zero: } \frac{1}{\tau}=\omega_m\sqrt{\beta}; \text{Pole: } \frac{1}{\beta\tau}.$$

- Draw the Bode plot of the system augmented by the compensator, check the resulting phase margin, and repeat the steps if necessary.

Compensator

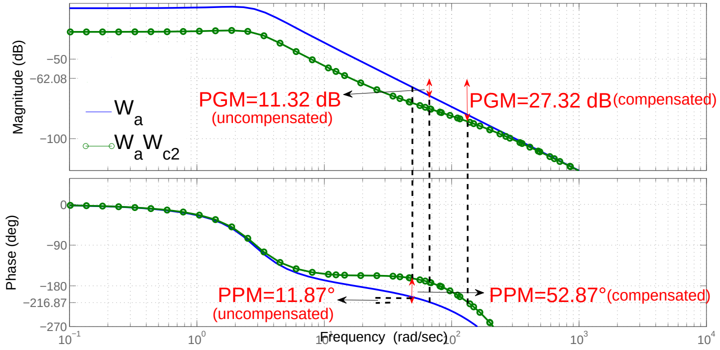


Figure: Uncompensated and compensated system $W_{c2}W_a = \frac{s+46.28}{s+154.27}\frac{e^{-0.01s}}{s^2+3s+8}$, 2-SMC

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