Conventional Sliding Mode design for Linear Systems

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Preliminaries

Consider

$$\dot{x} = Ax + B(u + f)$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $rank(B) = m$ and (A, B) controllable.

Conventional (Two steps) Sliding Mode design

- Selection of sliding manifold of order $n - m$ with desired zero dynamics.
- Construction of a sliding mode control ensuring the finite time convergence to selected sliding manifold and with theoretically exact compensation of perturbations.

Transformation of the system into regular form.

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{(n-m)\times m},\ B_2 \in \mathbb{R}^{m\times m}$$

with $det B_2 \neq 0$. The transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx, \qquad T = \begin{bmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & B_2^{-1} \end{bmatrix}$$

reduce the system to the regular form.

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + u + \tilde{f} \end{aligned}$$

(A, B) controllable $\implies$ (A_{11}, A_{12}) controllable.

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- In $s = 0$ the dynamics is reduced to

$$\dot{x}_1 = (A_{11} - A_{12}K)x_1$$

Standard Relay Sliding Mode Controller

The dynamics of S is given by

$$\dot{S} = A_{21}x_1 + A_{22}x_2 + K(A_{11}x_1 + A_{12}x_2) + u + \tilde{f}$$

denote

$$u_{eq_{nom}} = -A_{21}x_1 - A_{22}x_2 - K(A_{11}x_1 + A_{12}x_2).$$

Propose the controller

$$u = -\kappa(t,x)\mathrm{Sign}(\mathrm{S}),$$

where $\mathrm{Sign}(\mathrm{S})^T = \begin{bmatrix}\mathrm{Sign}(\mathrm{s}_1) & \dots & \mathrm{Sign}(\mathrm{s}_\mathrm{m})\end{bmatrix}$ and $\kappa(t,x) > 0$ and

Relay SMC without $u_{eq_{nom}}$ compensation,

$$V = \frac{1}{2} S^T S,$$

$$\dot{V} = S^T \left[-u_{eq_{nom}} - \kappa(t,x)\mathrm{Sign(S)} + \tilde{f} \right]$$

Choosing κ_i component wise

$$\gamma_1 = min_{t,x} \left(\kappa_i(t,x) - |\tilde{f}_i(t,x) - u_{i,eq_{nom}}| \right) \implies \dot{V} \leq -\gamma_1 V^{1/2},$$

Conclusions

- Compensation of the perturbation $\tilde{f}$
- solutions converges to $s = 0$ in finite time.
- Coordinates of $\kappa(t,x)$ can be selected component wise.

Relay SMC with $u_{eq_{nom}}$ compensation

$$u = u_{eq_{nom}} - \kappa(t,x)\mathrm{Sign(S)}, \quad V = \frac{1}{2}S^T S,$$

$$\dot{V} = S^T\left[-u_{eq_{nom}} + u_{eq_{nom}} - \kappa(t,x)\mathrm{Sign(S)} + \tilde{f}\right]$$

$$\dot{V} = S^T\left[\tilde{f} - \kappa(t,x)\mathrm{Sign(S)}\right]$$

$$\gamma_2 = min_{t,x}\left(\kappa_i(t,x) - |\tilde{f}_i(t,x)|\right) \implies \dot{V} \leq -\gamma_2 V^{1/2},$$

Conclusions

- Compensation of the perturbation $\tilde{f}$
- solutions converges to $s = 0$ in finite time.
- $\kappa_i(t,x)$ are selected component wise, without compensation of $u_{i,eq_{nom}}$.

Comparison with $u_{eq_{nom}}$ compensation and without it

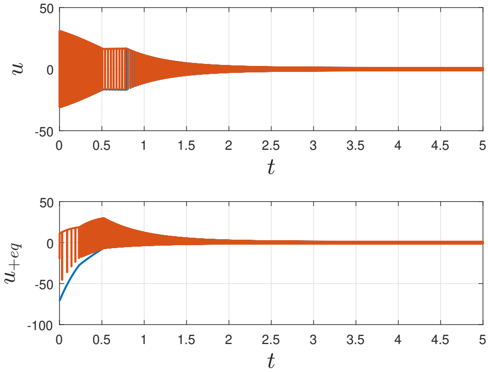

Figure: The upper plot is the control signal without compensation of $u_{eq_{nom}}$.

During the transition process, the chattering in control without compestion is bigger.

Unit Control

$$u = u_{eq_{nom}} - \rho(t,x)\frac{s}{\|s\|}$$

control u is discontinuous only in $s = 0$

$$\dot{V} = s^T\dot{s}, = s^T\tilde{f} - \rho(t,x)\|s\|,$$

$$\rho(t,x) > \|\tilde{f}(t,x)\| \implies \gamma_3 = min_{t,x}\left(\rho(t,x) - \|\tilde{f}(t,x)\|\right) \implies \dot{V} \le -\gamma_3 V^1$$

Consequently, $s = 0$ is reached in finite-time.

Conclusions

- $\rho(t,x)$ should compensate the norm of perturbations vector $\rightarrow$ Chattering will be bigger than in the relay controllers with component wise choose of the gains.

SMC without transformation into regular form

$$\dot{x} = Ax + B(u + f).$$

Switching Surface $s = Cx, \qquad C = [-K \quad I_m] \in \mathbb{R}^{m \times n} \quad det(CB) \neq 0$
Sliding mode dynamics $\dot{s} = CAx + CB(u + f),\ x \in R^n, u \in R^m$
Control Law
$u = u_{eq_{nom}} - \delta \mathrm{Sign(s)},\ u_{eq_{nom}} = -(CB)^{-1}CAx,\ \delta > max_{t,x}|f(t,x))|$

Conclusions

- Condition $CB + (CB)^T > 0$ is restrictive.

SMC without transformation into regular form

$$\dot{x} = Ax + B(u + f).$$

Desired surface $s = Cx$

CB is arbitrary matrix

Virtual sliding surface $s^* = (CB)s$

Control law $u = u_{eq_{nom}} - U(x)\mathrm{Sign}(s^*)$.

Lyapunov function $V = \frac{1}{2}{s^*}^T \dot{s}^*$,

$$\dot{V} = {s^*}^T s^* = {s^*}^T (CB)^{-1} CAxu_{eq_{nom}} - |s^*| U(x)$$

$$\leq |s^*| \cdot min_{x,t,i} |U_i(x) - f_i(x,t)|$$

then $\dot{V} < 0$ for $U_i(x)$ suitable.

$U_i(x)$ can be chosen component wise

Quadratic Minimization

- Quadratic performance index

$$J = \frac{1}{2}\int_{t_s}^{\infty} x^T(t)Qx(t)\,dt,$$

where Q is symmetric and positive definite, and t_s is the time at the sliding mode is reached.

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- Objetive: minimize the cost function under the assumption that sliding mode takes place.
- The cost function does not impose penalty cost on the control
- Represents a cost free control problem

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- the cos J in z coordinates

$$J = \frac{1}{2} \int_{t_s}^{\infty} z_1(t)^T Q_{11} z_1(t) + 2 z_1(t)^T Q_{12} z_2(t) + z_2(t)^T Q_{22} z_2(t)\, dt$$

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- If z_1 is consider as the state vector and z_2 as the virtual control input, then this expression represents a "traditional" mixed cost LQR problem

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- To avoid this complication define a new "virtual control" input

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- The cost function take the form

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where

$$\hat{Q} := Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T$$

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- Note that $\hat{Q}$ represents part of the Schur complement of matrix $T_r Q T_r^T$.

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- The positive definiteness of Q ensures from Shur complement arguments that $Q_{22} > 0$, so that Q_{22}^{-1} exists.

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- Riccati equation

$$\hat{P}\hat{A}^T + \hat{A}\hat{P} + \hat{Q} - \hat{P}A_{12}Q_{22}^{-1}A_{12}^T\hat{P} = 0$$

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- The matrix parameterizing the hyperplane is

$$M = Q_{22}^{-1}Q_{12}^T + Q_{22}^{-1}A_{12}^T\hat{P}$$