First Order Sliding Mode Existence and Stability

Leonid Fridman

Facultad de Ingeniería, UNAM

Sliding Mode Existence Conditions: Scalar Case

- Scalar Linear Control

$$\dot{s} = -s$$

$$s = 0$$

$$\lim_{s \to 0^+} \dot{s} = 0 \qquad \lim_{s \to 0^-} \dot{s} = 0$$

$$s\dot{s} \leq 0$$

It is not enough for Sliding Mode Existence !!!

- Relay Control

$$\dot{s} = -\mathrm{sign}(s)$$

$$s = 0$$

$$\lim_{s \to 0^+} \dot{s} = -1 \qquad \lim_{s \to 0^-} \dot{s} = 1$$

$$\lim_{s \to 0^+} \dot{s} < 0 \qquad \lim_{s \to 0^-} \dot{s} > 0$$

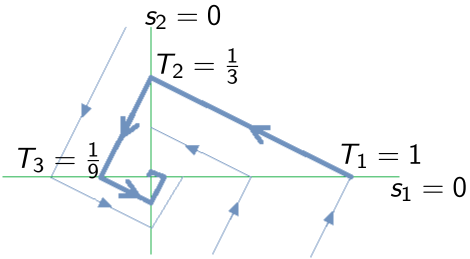
$$s\dot{s} \leq -|s|$$

Sufficient conditions for SM existence

Sliding Mode Existence Conditions

- Vector Control with Zeno Phenomena

$$\dot{s}_1 = -\mathrm{sign}(s_1) + 2\mathrm{sign}(s_2)$$
$$\dot{s}_2 = -2\mathrm{sign}(s_1) - \mathrm{sign}(s_2)$$



$$\sum_{i=1}^{\infty} T_i = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Piecewise Lyapunov Function I

Motion Equations:

$$\dot{s}_1 = -\mathrm{sign}(s_1) + 2\mathrm{sign}(s_2),$$
$$\dot{s}_2 = -2\mathrm{sign}(s_1) - \mathrm{sign}(s_2),$$

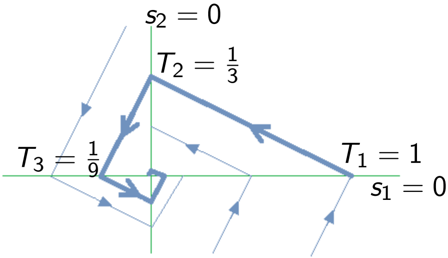
Lyapunov Function:

$$V = s^T \mathrm{sign}(s), \quad s^T = (s_1, s_2),$$

$$\mathrm{sign}(s) = \begin{bmatrix} \mathrm{sign}(s_1) \\ \mathrm{sign}(s_2) \end{bmatrix},$$

Time Derivative

$$\dot{V} = \frac{\partial V}{\partial s_1}\dot{s}_1 + \frac{\partial V}{\partial s_2}\dot{s}_2 = -2.$$



Sliding Mode Exists in $s = 0$.

Piecewise Lyapunov Function II

Motion Equations:

$$\begin{aligned}\dot{s}_1 &= -2\mathrm{sign}(s_1) - \mathrm{sign}(s_2),\\ \dot{s}_2 &= -2\mathrm{sign}(s_1) + \mathrm{sign}(s_2),\end{aligned}$$

Lyapunov Function:

$$V = s^T P \mathrm{sign}(s), \quad s^T = (s_1, s_2),$$

$$\mathrm{sign}(s) = \begin{bmatrix} \mathrm{sign}(s_1) \\ \mathrm{sign}(s_2) \end{bmatrix}, \qquad P = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix},$$

Time Derivative

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial s_1}\dot{s}_1 + \frac{\partial V}{\partial s_2}\dot{s}_2\\ &= -7 - 6\mathrm{sign}(s_1 s_2) < 0.\end{aligned}$$

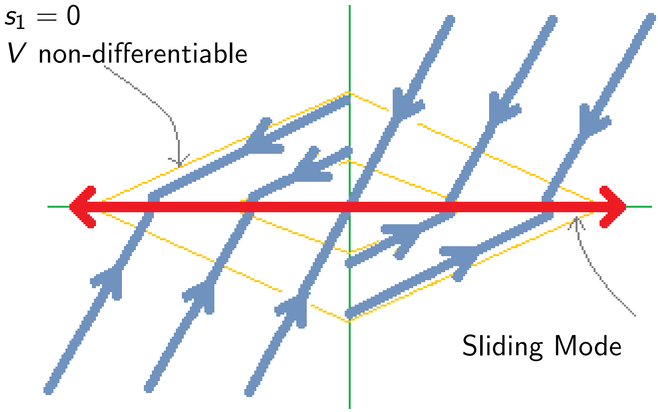
Sliding mode dynamics $s_1 = 0$

$$\begin{aligned}\dot{s}_1 &= u_{eq} - \mathrm{sign}(s_2),\\ \dot{s}_2 &= u_{eq} + \mathrm{sign}(s_2)\end{aligned}$$

$$\begin{aligned}u_{eq} &= sign(s_1),\\ \Longrightarrow \quad \dot{s}_2 &= 2\mathrm{sign}(s_2),\end{aligned}$$

$s_1 = 0 \implies s_2 \to \infty$ UNSTABLE!!

Sliding mode dynamics $s_1 = 0$



$s_1 = 0 \implies s_2 \to \infty$ UNSTABLE!!

Synthesis of Extended Invariance Principle for DHRS

Equation with Discontinuous Right Hand Side

$$\dot{x} = \phi(x), \quad x \in \mathbb{R}^n. \quad (DRHS)$$

Synthesis of Extended Invariance Principle for DHRS

- $V(x) > 0$ Lipschitz continuous Lyapunov function.
- $V(x)$ non-differentiable in $\mathcal{S} \in \mathbb{R}^n$.
- $\dot{V} \leq 0 \quad \forall\, x \in \{\mathbb{R}^n \setminus \mathcal{S}\}$
- $\dot{V} = 0 \quad \forall\, x \in \mathcal{W} \in \mathbb{R}^n$
- Origin of $(DRHS)$ for dynamics on $\mathcal{S}$ is stable if they exist.
- Origin of $(DRHS)$ for dynamics on $\mathcal{W}$ is stable if they exist.

(Orlov, TAC 2003)

Origin of $(DRHS)$ is stable.

Extended Invariance Principle (Orlov, 2003)

- Stability in subspace $\mathcal{S}$: $\dot{s} = Gf + GBu$
- Equivalent to Equation: $\dot{s} = d(x) - \alpha D(x)\mathrm{sign(s)}$
- $[\mathrm{sign(s)}]^T = [\mathrm{sign(s_1)}, \ldots, \mathrm{sign(s_m)}]$

Theorem

If $D + D^T > 0$ then there exists $\alpha_0 > 0$ such that sliding mode exists in manifold $s = 0$ for $\alpha > \alpha_0$.

The statement of the theorem may be proven using sum of absolute values of s_i

$$V = s^T \mathrm{sign(s)} > 0$$

as Lipschitz(not differentiable!) Lyapunov function.

Unit Control [Gutman and Leitmann, 1976]

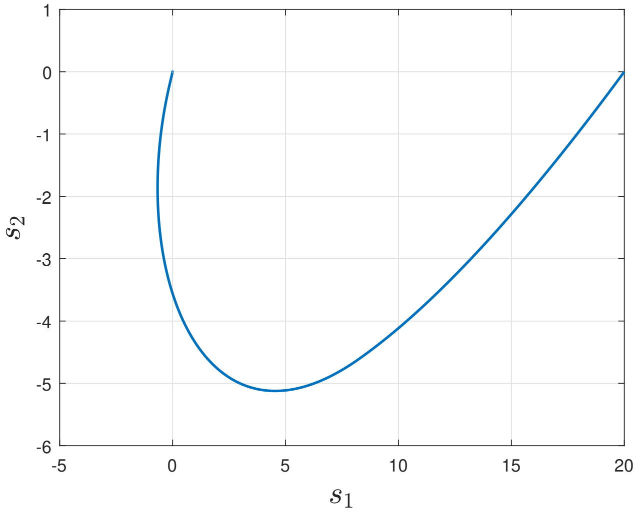
$$\dot{s}_1 = \frac{-s_1}{\sqrt{s_1^2 + s_2^2}},$$

$$\dot{s}_2 = \frac{-s_2}{\sqrt{s_1^2 + s_2^2}}$$

Advantages

- DRHS is discontinuous only in origin.
- Solutions reach origin $s_1 = s_2 = 0$ in finite time.

Unit Control phase plane



Unit Control [Gutman and Leitmann, 1976]

$$\dot{s} = Du + d(x), \quad D^T + D > 0, \quad \lambda_{min}(D^T + D) - 2\|d(x)\| > \gamma > 0.$$

Unit Controller

$$u = -\frac{s}{||s||},$$

Quadratic Lyapunov function $V = s^T s = \|s\|^2$

$$\dot{V} = -s^T \left(D\frac{s}{||s||} + d(x) \right) - \left(\frac{s}{||s||} D + d(x) \right)^T s,$$
$$\leq -\left(\lambda_{min}(D^T + D) - 2\|d(x)\| \right) ||s|| \leq -\gamma\sqrt{V}$$

Unit Control [Gutman and Leitmann, 1976]

Finite time stability

$$\dot{V} < -\gamma\sqrt{V} \implies \frac{d\dot{V}}{V} < -\gamma \implies 2\sqrt{V} \leq 2\sqrt{V(s_0)} - \gamma t$$

$$T_{es} \leq \frac{2}{\gamma}\sqrt{V(s_0)}$$

Disadvantages

- Difficult to implement in computer simulations
- Gains of the control should be much bigger since it compensates perturbations in every channel
- If one channel fail, all the controller fails

Invariance of Sliding Modes

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t)$$

- $h(x,t)$ is disturbance vector
- $h(x,t) \in range(B)$
- Matching Condition

B. Drazenovic,
The invariance conditions in variable structure systems, *Automatica*, v.5, No.3,Pergamon Press, 1969.

Design of Control Under Uncertainty Conditions

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t),$$

- Invariance condition

$$h(x,t) \in range(B),$$
$$\exists \lambda(x,t): \quad h = B\lambda, \quad \lambda \in \mathbb{R}, \quad then$$
$$\dot{x} = f(x,t) + B(x,t)(u+\lambda), \qquad (*)$$

$s(x) = 0$ is a sliding manifold

Design of Control Under Uncertainty Conditions

$$\dot{x} = f(x,t) + B(x,t)u + h(x,t),$$

- Invariance condition

$$\begin{gathered} h(x,t) \in range(B), \\ \exists \lambda(x,t): \quad h = B\lambda, \quad \lambda \in \mathbb{R}, \quad then \\ \dot{x} = f(x,t) + B(x,t)(u+\lambda), \qquad (*) \end{gathered}$$

$s(x) = 0$ is a sliding manifold

- Equivalent Control Method:

$$\begin{gathered} \dot{s} = 0 \implies u_{eq}: \\ \dot{s} = Gf + GB(u_{eq} + \lambda) = 0, \quad G := \left\{ \frac{\partial s}{\partial x} \right\} \\ u_{eq} + \lambda = -(GB)^{-1} Gf \implies (*) \\ (*) \implies \dot{x} = f(x,t) - B(x,t)(GB)^{-1} Gf \end{gathered}$$

does not depend on disturbance $h(x,t)$.

Design of Control Under Uncertainty Conditions

- However, if

$$\begin{aligned}
&\exists\lambda(x,t): \quad h = B\lambda_1 + B^{\perp}\lambda_2,\\
&\lambda_1,\lambda_2 \in \mathbb{R}, \quad \textit{then}\\
&\dot{x} = f(x,t) + B(x,t)(u+\lambda_1) + B^{\perp}\lambda_2,
\end{aligned}$$

$s(x) = 0$ is a sliding manifold

- Equivalent Control Method:

$$\begin{aligned}
&\dot{s} = 0 \implies u_{eq}:\\
&\dot{s} = Gf + GB(u_{eq} + \lambda_1) + B^{\perp}\lambda_2 = 0,\\
&G := \left\{\frac{\partial s}{\partial x}\right\}\\
&u_{eq} + \lambda_1 = -(GB)^{-1}[Gf + B^{\perp}\lambda_2]\\
&\dot{x} = f(x,t) - B(x,t)(GB)^{-1}[Gf + B^{\perp}\lambda_2].
\end{aligned}$$

In this case invariance does not exists.

Output control for a system with relative degree 2 with unmatched perturbations

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1, \quad f_1 \neq 0 \\ \dot{x}_2 &= u + f_2. \end{aligned}$$

if x_1 is the output, then

$$\ddot{x}_1 = u + f_2 + \dot{f}_1,$$

Perturbations are matched with respect to the output x_1.