Rated Stability

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UNAM

Outline

1. Introduction
   - Historical Remarks
   - Model of Dynamical System

2. Stability Notions
   - Unrated Stability
   - Rated Stability
   - Non-Asymptotic Stability

A.M. Lyapunov (1857-1918) and the first page of his thesis

ОБЩАЯ ЗАДАЧА

ОБЪ

УСТОЙЧИВОСТИ ДВИЖЕНІЯ.

РАЗСУЖДЕНІЕ

А. ЛЯПУНОВА.

Изданіе Харьковскаго Математическаго Общества.

ХАРЬКОВЪ.

Типографія Зильберберга, Рыбная ул., д. № 25-й

1892.

Pendulum Equation

Consider the pendulum equation

$$\ddot{\theta}(t) + k\dot{\theta}(t) + \frac{g}{r}\sin\left(\theta(t)\right) = 0$$

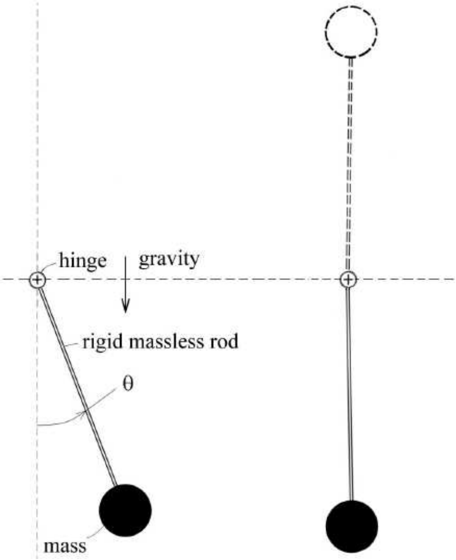
where

θ – an inclination angle,

k – a friction coefficient

r – a length of pendulum,

g – the gravitation constant.



Outline

Historical Remarks

Unrated Stability

Lyapunov Stability, Asymptotic Stability
(Lyapunov 1892, Zubov 1957, Krasovskii 1959, LaSalle & Lefschetz 1960, Hahn 1961, Roxin 1965 etc)

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Rated Stability

Exponential, Finite-time and Fixed-time Stability
(Erugin 1951, Zubov 1957, Hahn 1961, Roxin 1966, Demidovich 1974, Bhat & Bernstein 2000, Orlov 2005, Levant 2005, Moulay & Perruquetti 2005, Andrieu et al 2008,Cuz, Moreno, Fridman 2010, Polyakov 2012,...)

Outline

System Description

Model of the System

Consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad t \in \mathbb{R}; \tag{Sys}$$

$$x(t_0) = x_0, \quad x_0 \in \mathbb{R} \tag{IC}$$

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Assumption

$$0 \in F(t, 0) \quad \text{for} \quad t \in \mathbb{R}$$

Notation

$\Phi(t, t_0, x_0)$– **Set of all solutions** of the Cauchy problem (Sys);
$x(t, t_0, x_0) \in \Phi(t, t_0, x_0)$– **a solution** of (Sys)-(IC).

Weak Stability

Example

Weakly stable system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 \in -\left(k_1 \overline{\text{sign}}[x_1] \dot{+} k_2 \overline{\text{sign}}[x_2]\right), \end{cases}, \quad x_i \in \mathbb{R},$$

Weak Stability

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$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 \in -\left(k_1\overline{\text{sign}}[x_1] + k_2\overline{\text{sign}}[x_2]\right), \end{cases}, \quad x_i \in \mathbb{R},$$

2 cases

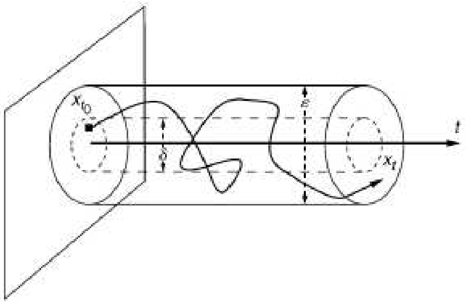
- If $k_1 > k_2 > 0 \quad \rightarrow \quad x_1 = 0, x_2 = 0$ is finite stable equilibrium point
- If $k_2 > |k_1| \quad \rightarrow \quad x_1(t) = constant, x_2 = 0$ is a solution.

Outline

Definition (Stability, Lyapunov 1892)

The origin of the system (Sys) is said to be **Lyapunov stable** if $\forall \epsilon \in \mathbb{R}_+$ and $\forall t_0 \in R : \exists \delta = \delta(\epsilon, t_0) \in \mathbb{R}_+$ such that $\forall x_0 \in \mathbb{B}(\delta)$

1. any solution $x(t, t_0, x_0)$ of Cauchy problem (Sys)-(IC) exists for $t > t_0$;
2. $x(t, t_0, x_0) \in \mathbb{B}(\epsilon)$ for $t > t_0$.



Uniform Stability and Instability

Definition (Uniform Lyapunov Stability)

If the function δ in Definition of Lyapunov Stability does not depend on t_0 then the origin is called **uniformly Lyapunov stable**.

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Proposition

If the origin of the system (Sys) *is Lyapunov stable then* $x(t) = 0$ *is the unique solution of Cauchy problem* (Sys)-(IC) *with* $x_0 = 0$ *and* $t_0 \in \mathbb{R}$.

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Definition (Instability)

The origin, which does not satisfy any condition from Lyapunov Stability definition, is called **unstable**.

Example (Uniformly Lyapunov Stable System)

$$\begin{cases} \dot{x}_1 \in \overline{\text{sign}}[-x_2], \\ \dot{x}_2 \in \overline{\text{sign}}[x_1] \end{cases}, \quad x_1, x_2 \in \mathbb{R}$$

Example (Uniformly Lyapunov Stable System)

No sliding motion

$$\begin{cases} \dot{x}_1 = \text{sign}[-x_2], \\ \dot{x}_2 = \text{sign}[x_1] \end{cases}, \quad x_1, x_2 \in \mathbb{R}$$

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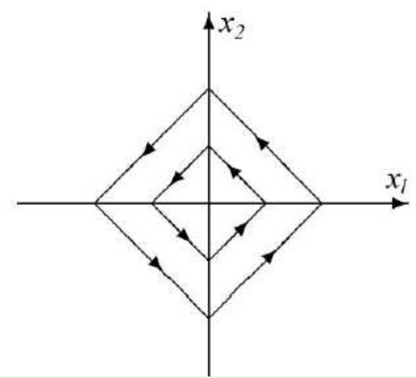

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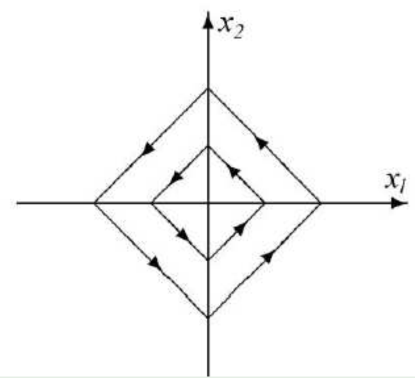
$$\dot{V} = sign(x_1)\dot{x}_1 + sign(x_2)\dot{x}_2 = 0$$



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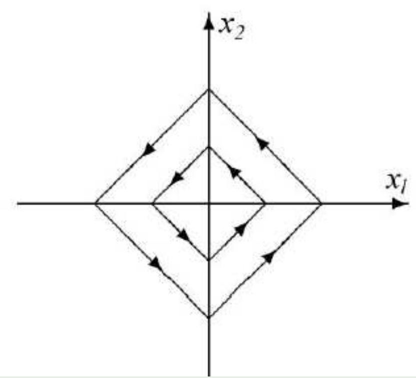
Lyapunov Stability



Definition (Asymptotic attrativity)

The origin of the system (Sys) is said to be asymptotically attractive if $\forall t_0 \in \mathbb{R}$ exists a set $\mathbb{U}(t_0) \subseteq \mathbb{R}^n : \mathbb{U}(t_0) \setminus 0$ is non-empty and $\forall x_0 \in \mathbb{U}(t_0)$

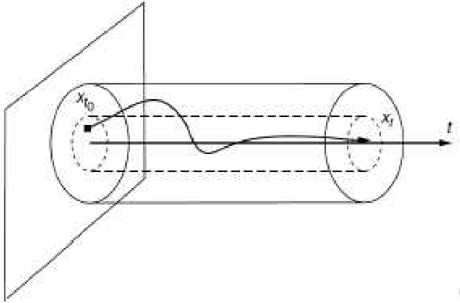
- any solution $x(t, t_0, x_0)$ of Cauchy problem (Sys)-(IC) exists for $t > t_0$;
- $\lim_{t \to +\infty} \|x(t, t_0, x_0)\| = 0$.

The set $\mathbb{U}(t_0)$ is called attraction domain.

Definition (Asymptotic stability)

The origin of the system (Sys) is said to be asymptotically stable if it is

- Lyapunov stable;
- asymptotically attractive with an attraction domain $\mathbb{U}(t_0) \subseteq \mathbb{R}^n$ such that $0 \in \text{int}(\mathbb{U}(t_0))$ for all $t_0 \in \mathbb{R}$.



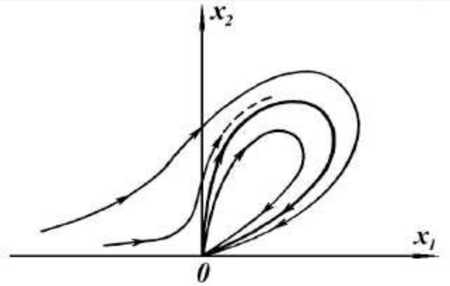
Asymptotic attractivity may not imply Asymptotic Stability

Example (R.E. Vinograd 1957)

$$\dot{x}_1 = \frac{x_1^2 (x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)}$$

and

$$\dot{x}_2 = \frac{x_2^2 (x_2 - 2x_1)}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)}$$



Definition (Uniform Asymptotic Attractivity)

The origin of the system (Sys) is said to be uniformly asymptotically attractive

- if it is asymptotically attractive with a time-invariant attraction domain $\mathbb{U} \subseteq \mathbb{R}^n$;
- $\forall R \in \mathbb{R}_+$, $\forall \epsilon \in \mathbb{R}_+$ there exists $T = T(R, \epsilon) \in \mathbb{R}_+$ such that the inclusions $x_0 \in \mathbb{B}(R) \cap \mathbb{U}$ and $t_0 \in \mathbb{R}$ imply $x(t, t_0, x_0) \in \mathbb{B}(\epsilon)$ for $t > t_0 + T$.

Uniform Asymptotic Stability

Definition (Uniform asymptotic stability)

The origin of the system (Sys) is said to be **uniformly asymptotically stable** if it is *uniformly Lyapunov stable* and *uniformly asymptotically attractive* with an attraction domain $\mathbb{U} \subseteq \mathbb{R}^n : 0 \in \text{int}(\mathbb{U})$.

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Proposition (Clarke, Ledyaev, Stern 1998)

Let a set-valued function $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined and upper-semicontinuous in R^n. Let $F(x)$ be nonempty, compact and convex for any $x \in \mathbb{R}^n$.
If the origin of the system

$$\dot{x} \in F(x)$$

is asymptotically stable then it is **uniformly** *asymptotically stable.*

Outline

Definition (Exponential stability)

The origin of the system (Sys) is said to be exponentially stable if there exist an attraction domain $\mathbb{U} \subseteq \mathbb{R}^n : 0 \in \text{int}(\mathbb{U})$ and numbers $C, r \in \mathbb{R}_+$ such that

$$\|x(t, t_0, x_0)\| \leq C\|x_0\|e^{-r(t-t_0)}, t > t_0.$$

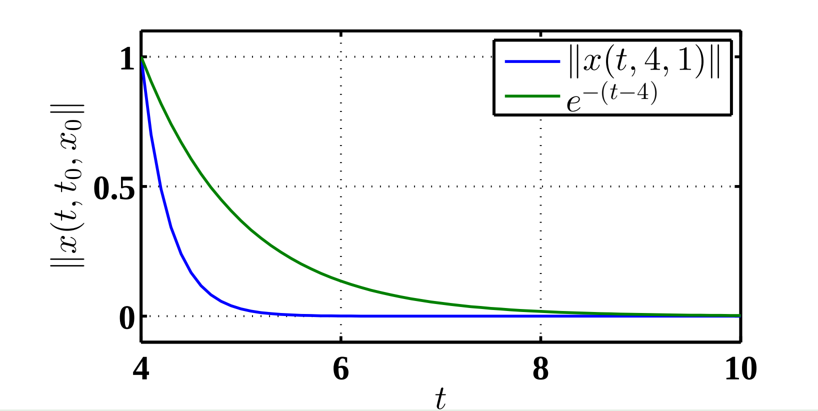
for $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{U}$.

The theory of Linear Control Systems deals with exponential stability

Example (Linear Variable Structure System)

$$\dot{x} = -(2 - \text{sign}[\sin(x)])x, \quad x \in \mathbb{R}, \quad x(t_0) = x_0$$

$$\|x(t, t_0, x_0\| \leq |x_0| e^{-(t-t_0)}, \quad t > t_0$$



Example (Homogeneous system)

$$\dot{x} = -\lfloor x \rceil^{\alpha}; \quad x(0) = x_0 \;\; 0 < \alpha < 1$$ [1]

[1] $\lfloor \cdot \rceil^{\alpha} = |\cdot|^{\alpha}\,\mathrm{sign}[\cdot]$

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$$\dot{x} = -\lfloor x \rceil^{\alpha}; \quad x(0) = x_0 \quad 0 < \alpha < 1$$ [1]

$$x_0 >= 0$$

$$\Rightarrow x(t) >= 0 \quad \forall t \in [0, t_1]$$

$$\dot{x} = -x^{\alpha}$$

$$\Rightarrow x(t) = \left(x_0^{1-\alpha} - (1-\alpha)t\right)^{\frac{1}{1-\alpha}}$$

$$x(t) = 0 \text{ at } \quad t = \frac{x_0^{1-\alpha}}{1-\alpha}$$

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$$x_0 <= 0$$

$$\Rightarrow x(t) <= 0 \quad \forall t \in [0, t_1]$$

$$\dot{x} = (-x)^{\alpha}$$

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$$x(t) = 0 \text{ at } \quad t = \frac{(-x_0)^{1-\alpha}}{1-\alpha}$$

$$\therefore \quad x(t) = 0 \quad \forall t \geq \frac{|x_0|^{1-\alpha}}{1-\alpha}$$

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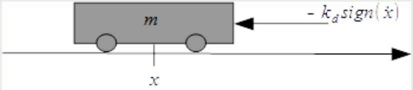
Mechanical Example

Example (Deceleration of a Cart)



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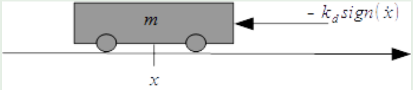

$$m\ddot{x} = -k_d \operatorname{sign}[\dot{x}]$$

m – mass

x – position

k_d – coefficients of dry friction

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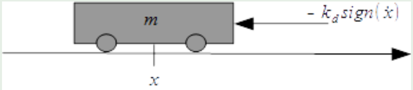

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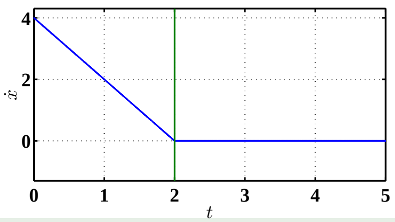

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Finite-time attractivity

Introduce the functional $T_0 : \mathbb{W}^n_{[t_0,+\infty)} \to \mathbb{R}_+ \cup \{0\}$ by

$$T_0(y(\cdot)) = \inf_{\tau \geq t_0 : y(\tau)=0} \tau.$$

If $y(\tau) \neq 0$ for all $t \in [t_0, +\infty)$ then $T_0(y(\cdot)) = +\infty$.
Define the **settling-time function** of the system (Sys) as

$$T(t_0, x_0) = \sup_{x(t,t_0,x_0) \in \Phi(t_0,x_0)} T_0(x(t, t_0, x_0)) - t_0.$$

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Definition (Finite-time attractivity)

The origin of the system (Sys) is said to be finite-time attractive if $\forall t_0 \in \mathbb{R}$ exists a set $\mathbb{V}(t_0) \subseteq \mathbb{R}^n : V(t_0) \setminus \{0\}$ is non-empty and $\forall x_0 \in \mathbb{V}(t_0)$

- any solution $x(t, t_0, x_0)$ of Cauchy problem (Sys)-(IC) exists for $t > t_0$;
- $T(t_0, x_0) < +\infty$ for $x_0 \in \mathbb{V}(t_0)$ and for $t_0 \in \mathbb{R}$.

The set $\mathbb{V}(t_0)$ is called **finite-time attraction domain**.

Finite-time Stability (Erugin 1991, Zubov 1957, etc)

Definition (Finite-time stability, Roxin 1966)

The origin of the system (Sys) is said to be finite-time stable if it is Lyapunov stable and finite-time attractive with an attraction domain $\mathbb{V}(t_0) \subseteq \mathbb{R}^n$ such that $0 \in \text{int}(\mathbb{V}(t_0))$ for any $t_0 \in \mathbb{R}$.

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Proposition (Bhat & Bernstein 2000)

If the origin of the system (Sys) *is finite-time stable then it is asymptotically stable and* $x(t, t_0, x_0) = 0$ *for* $t > t_0 + T_0(t_0, x_0)$.

Uniform Finite-time Stability (Roxin 1966, Praly 1997, etc)

Definition (Uniform finite-time attractivity)

The origin of the system (Sys) is said to be uniformly finite-time attractive if

- it is finite-time attractive with a time-invariant attraction domain $\mathbb{V} \subseteq R^n$;
- $T(t_0, x_0)$ is locally bounded on $\mathbb{R} \times \mathbb{V}$ uniformly on $t_0 \in \mathbb{R}$, i.e.

$$\forall y \in \mathbb{V} : \exists \epsilon \in \mathbb{R}_+ \Rightarrow \sup_{\substack{t_0 \in \mathbb{R}, \\ x_0 \in \{y\} \dotplus \mathbb{B}(\epsilon) \subset \mathbb{V}}} T(t_0, x_0) < +\infty.$$

Uniform Finite-time Stability (Roxin 1966, Praly 1997, etc)

Definition (Uniform finite-time stability (Orlov 2005)))

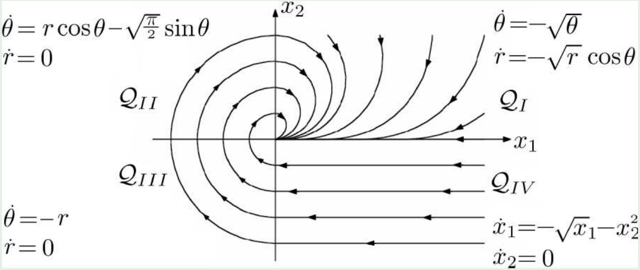
The origin of the system (Sys) is said to be uniformly finite-time stable if it is uniformly Lyapunov stable and uniformly finite-time attractive with an attraction domain $\mathbb{V} \subseteq \mathbb{R}^n : 0 \in \mathrm{int}(\mathbb{V})$.

Example

$$\dot{x} \in -\overline{\mathrm{sign}}[x], \quad x \in \mathbb{R}, \quad T(t_0, x_0) = |x_0|$$

Time invariance does not imply uniformity of FTS

Example (S.P. Bhat & D. Bernstein 2000)



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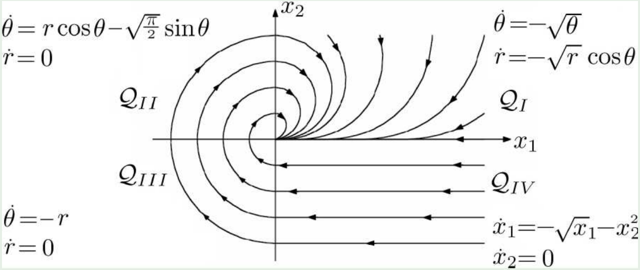

Denote $x_0^i = \left(0, \frac{-1}{i}\right)^T$, $i = 1, 2, 3, \ldots$

$$x_0^i \to 0 \quad \text{and} \quad T(x_0^i) \to +\infty$$

Example

Two uniformly finite-time stable systems Consider two systems[2]

$$(I) \quad \dot{x} = -\lfloor x \rceil^{\frac{1}{2}} (1 - |x|), \qquad (II) \quad \dot{x} = \begin{cases} -\lfloor x \rceil^{\frac{1}{2}} & \text{for} \quad x < 1, \\ 0 & \text{for} \quad x \geq 1, \end{cases}$$

which are uniformly finite-time stable with $\mathbb{V} = \mathbb{B}(1)$.

[2] $\lfloor x \rceil^{[\rho]} = |x|^{\rho} \operatorname{sign}[x]$

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which are uniformly finite-time stable with $\mathbb{V} = \mathbb{B}(1)$.

$$T_{(I)}(x_0) = \ln \left(\frac{1 + |x_0|^{\frac{1}{2}}}{1 - |x_0|^{\frac{1}{2}}} \right) \qquad T_{(II)}(x_0) = 2|x_0|^{\frac{1}{2}}.$$

$$T_{(I)}(x_0) \to +\infty \quad \text{if} \quad x_0 \to \pm 1 \qquad T_{(II)}(x_0) \to 2 \quad \text{if} \quad x_0 \to \pm 1$$

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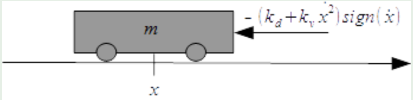
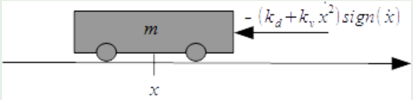


Mechanical Example

Example (Deceleration of a Cart)



$$m\ddot{x} = -\left(k_d + k_v\dot{x}^2\right)\operatorname{sign}[\dot{x}], \quad t > 0$$

m – mass

x – position

k_d, k_v – coefficients of dry and viscous friction

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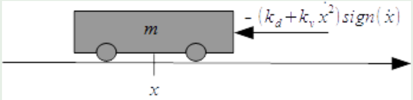

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$$\dot{x}(t) = 0, \quad \forall t \geq T_{max} = \frac{m\pi}{2\sqrt{k_d k_v}} \quad \text{and for any} \quad (x_0, \dot{x}_0) \in \mathbb{R}^2$$

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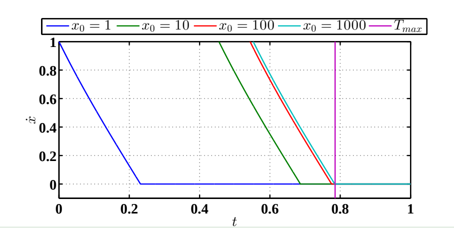

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Example

$$\dot{x} = -\lfloor x \rceil^{\frac{1}{2}} - \lfloor x \rceil^{\frac{3}{2}}, \quad x(0) = x_0 > 0$$

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$$2\arctan\sqrt{x} = 2\arctan\sqrt{x_0} - t, \quad \forall x_0, \quad t < \pi$$

Fixed-time Stability(Balakrishan 1996, Andrieu et al 2008,Cruz, Moreno, Fridman 2010,...)

Definition (Fixed-time attractivity)

The origin of the system (Sys) is said to be fixed-time attractive if

- it is uniformly finite-time attractive with an attraction domain $\mathbb{V}$;
- $T(t_0, x_0)$ is bounded on $\mathbb{R} \times \mathbb{V}$, i.e.

$$\exists T_{max} \in \mathbb{R}_+ \ \text{ such that } \ T(t_0, x_0) \leq T_{max} \ \text{ if } \ t_0 \in \mathbb{R}, x_0 \in \mathbb{V}$$

Definition (Fixed-time stability, Polyakov 2012)

The origin of the system (Sys) is said to be fixed-time stable if it is Lyapunov stable and fixed-time attractive with an attraction domain $\mathbb{V} \subseteq \mathbb{R}^n : 0 \in \text{int}(\mathbb{V})$.

Fixed-time stability $\Rightarrow$ NON-Asymptotic estimation & control

$$\begin{cases} \dot{x}(t) = u(t) \\ x(t) = x_0, \end{cases} \quad x, u \in \mathbb{R}$$

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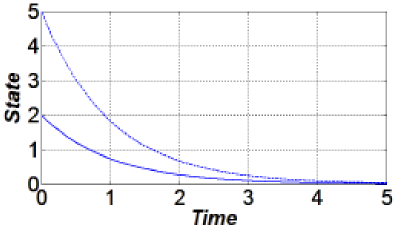
Asymptotic stabilisation:

$$u(t) = -x(t)$$

$$x(t) = e^{-t}x_0 \to 0 \quad \text{if} \quad t \to +\infty$$



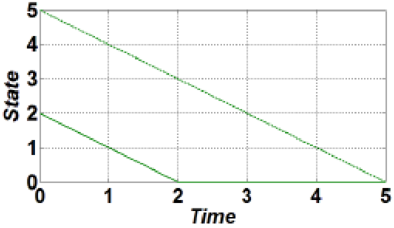
Fixed-time stability ⇒ NON-Asymptotic estimation & control

$$\begin{cases} \dot{x}(t) = u(t) \\ x(t) = x_0, \end{cases} \qquad x, u \in \mathbb{R}$$

Finite-Time stabilisation:

$$u(t) = -\lfloor x(t) \rceil^0$$

$$x(t) = 0 \quad \text{for} \quad t \geq \|x_0\|$$



Fixed-time stability $\Rightarrow$ NON-Asymptotic estimation & control

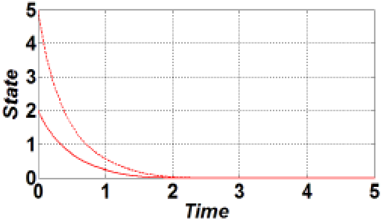
$$\begin{cases} \dot{x}(t) = u(t) \\ x(t) = x_0, \end{cases} \qquad x, u \in \mathbb{R}$$

Fixed-Time stabilisation:

$$u(t) = -\lfloor x(t) \rceil^{\frac{1}{2}} - \lfloor x(t) \rceil^{\frac{3}{2}}$$

$$x(t) = 0 \quad \text{for} \quad t \geq \pi$$

independently of x_0



Summary

- In general case, attractivity does not imply stability.
- Strong stability is more preferable for control applications.
- Control theory is mainly interested in rated stability.