Equations with Discontinuous Right Hand Side

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UNAM
Outline

1 Preliminaries
   - Absolute Continuity
   - Upper semi-continuity

2 Equations with Discontinuous RHS
   - Historical Remarks

3 Regularization Procedure for ODE with Discontinuous RHS
   - Filippov Solutions
   - Utkin Solutions
   - Aizerman-Pyatniskii Solutions

4 Disturbed Systems and Extended Differential Inclusions
   - Disturbances and Differential Inclusions

5 Existence of Solutions
   - Existence Conditions
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Absolute Continuity

Definition

Let $\mathcal{I}$ be an interval in the real line $\mathbb{R}$. A function $f : \mathcal{I} \to \mathbb{R}$ is absolutely continuous on $\mathcal{I}$ if for every positive number $\epsilon$, there is a positive number $\delta$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k; y_k)$ of $\mathcal{I}$ satisfies

$$\sum_{k} (y_k - x_k) < \delta$$

then

$$\sum_{k} |f(y_k) - f(x_k)| < \epsilon$$

The collection of all absolutely continuous functions on $\mathcal{I}$ is denoted $AC(\mathcal{I})$. 
equivalent definitions

1. $f$ is absolutely continuous

2. $f$ has a Lebesgue integrable derivative $f'$ almost everywhere and

$$f(x) = f(a) + \int_{a}^{x} f'(t) dt; \quad \forall x \in [a; b]$$

3. there exists a Lebesgue integrable function $g$ on $[a; b]$ such that

$$f(x) = f(a) + \int_{a}^{x} g(t) dt; \quad \forall x \in [a; b]$$

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental theorem of Lebesgue integral calculus, due to Lebesgue.
# Absolute continuity of functions

## Properties

1. If \( f, g \in AC(I) \), then \( f \pm g \) is absolutely continuous.
2. If \( I \) is a bounded closed interval and \( f, g \in AC(I) \), then \( fg \) is also absolutely continuous.
3. If \( I \) is a bounded closed interval, \( f \in AC(I) \) and \( f \neq 0 \) then \( \frac{1}{f} \) is absolutely continuous.
4. Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
5. If \( f : I \to \mathbb{R} \) is absolutely continuous, then it is of bounded variation on \( [a; b] \).
Example

\[ f(x) = \begin{cases} 
0 & \text{if } x = 0 \\
 x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 
\end{cases} \]

on a finite interval containing the origin.
Example

\[ f(x) = |x|^{1/2} \]

in zero it is not differentiable and the lateral derivatives do not exist.
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Introduce the following distances

\[
\rho(x, M) = \inf_{y \in M} \|x - y\|, \quad x \in \mathbb{R}^n, M \subset \mathbb{R}^n,
\]

\[
\rho(M_1, M_2) = \sup_{x \in M_1} \rho(x, M_2), \quad M_1 \subset \mathbb{R}^n, M_2 \subset \mathbb{R}^n,
\]

In general, the distance \( \rho \) is not symmetric, \( \rho(M_1, M_2) \neq \rho(M_2, M_1) \).
Introduce the following distances

\[ \rho(x, M) = \inf_{y \in M} \| x - y \|, \quad x \in \mathbb{R}^n, M \subset \mathbb{R}^n, \]

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Definition

A set-valued function $F : \mathbb{R}^{n+1} \to 2^{\mathbb{R}^{n+1}}$ is said to be **upper semi-continuous** at a point $(t^*, x^*) \in \mathbb{R}^{n+1}$ if $(t, x) \to (t^*, x^*)$ implies

$$\rho(F(t, x), F(t^*, x^*)) \to 0.$$
Example (Upper semi-continuous set-valued function)

\[
\text{sign}[\rho] = \begin{cases} 
1 & \text{if } \rho > 0 \\
-1 & \text{if } \rho < 0 \\
[-1, 1] & \text{if } \rho = 0 
\end{cases}
\]

is an upper semi-continuous set-valued function.
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Historical Remarks

Differential Equations with Discontinuous RHS

\[ \dot{x}(t) = f(t, x(t)), \]
\[ t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \]

- RHS Discontinuous with respect to the **time** variable
  (Caratheodory 1927)

Constantin Caratheodory
(1873-1950)
Differential Inclusions (Contingent Differential Equations)

\[ \dot{x}(t) \in F(t, x(t)), \]
\[ t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad F : \mathbb{R} \times \mathbb{R} \rightarrow 2^{\mathbb{R}^n} \]


Stanislaw Zaremba
(1863-1942)
Historical Remarks

Differential Equations with Discontinuous RHS

\[ \dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \]

- RHS Discontinuous with respect to the state variable (Filippov 1960, Utkin 1967, Aizerman & Pyatnitskii 1974)

Professors A. Filippov, E. Pyatnitskii, M. Aizerman and V. Utkin
\[ \dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n \] (DiscRHS)
ODE with Discontinuous RHS

\[ \dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}, \; x \in \mathbb{R}^n, \; f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n \]  

(DiscRHS)

\( f \) is piecewise continuous:

1. \( \mathbb{R}^{n+1} = \bigcup_{j=1}^{N} \tilde{G}_j \), where \( \tilde{G}_j \)-open connected sets \( G_i \cap G_j \neq \emptyset, \; i \neq j \);

2. \( \mathcal{S} \bigcup_{j=1}^{N} \partial G_j \) is of measure zero;

3. \( f(t, x) \) is continuous in each \( G_j \) and \( \forall (t, x) \in \partial G_j : \exists f^j(t, x) \in \mathbb{R}^n \)

\[ f^j = \lim_{(t^k, x^k) \to (t, x)} f(t^k, x^k), \]

\( (t^k, x^k) \in G_j, \; (t, x) \in \partial G_j \)
Filippov Regularization

\[ \dot{x}(t) \in F(t, x(t)), \quad t \in \mathbb{R} \quad \text{(DiffInc)} \]

\[ F(t, x) = \begin{cases} 
\{ f(t, x) \} & \text{if } (t, x) \in \mathbb{R}^{n+1} \setminus S, \\
\co \left( \bigcup_{j \in \mathcal{N}(t, x)} \{ f^j(t, x) \} \right) & \text{if } (t, x) \in S,
\end{cases} \]

\[ \mathcal{N}(t, x) = \{ j \in \{1, 2, \ldots, N\} : (t, x) \in \partial G_j \}. \]
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\end{cases}
\]

\[
\mathcal{N}(t, x) = \{j \in \{1, 2, \ldots, N\} : (t, x) \in \partial G_j\}.
\]

Definition (Filippov 1960)

An absolutely continuous function \(x : \mathcal{I} \to \mathbb{R}^n\) defined on some interval or segment \(\mathcal{I}\) is called a solution of (DiscRHS) if it satisfies the differential inclusion (DiffInc) almost everywhere on \(\mathcal{I}\).
Illustration of Filippov regularization

\[ \dot{x}(t) - [x(t)] + d(t), \quad t > 0, \]

where \( x(\cdot) \in \mathbb{R}, \parallel d \parallel_{C} \leq d_0 < 1. \)

\[ \rho = \begin{cases} 1 & \rho > 0 \\ -1 & \rho < 0 \\ 0 & \rho = 0 \end{cases} \]

(a) Switching case. (b) Sliding mode.
Illustration of Filippov regularization

Example

\[ \dot{x}(t) = - \text{sign} [x(t)] + d(t), \quad t > 0, \]

where \( x(\cdot) \in \mathbb{R}, \|d\|_C \leq d_0 < 1 \).

\[
\text{sign} [\rho] = \begin{cases} 
1 & \text{if } \rho > 0 \\
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5 Existence of Solutions
   - Existence Conditions
Let us consider the system

\[ \dot{x}(t) = f(t, x(t), u(t, x(t))), \quad t \in \mathbb{R}, \]  

(DisContSys)

where \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \ f \in C \) and

\[ u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m, \quad u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_m(t, x))^T \]

is a **piecewise continuous** feedback control.
Discontinuous Control Systems

Let us consider the system

$$\dot{x}(t) = f(t, x(t), u(t, x(t))), \quad t \in \mathbb{R},$$

(DisContSys)

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $f \in C$ and

$$u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_m(t, x))^T$$

is a **piecewise continuous** feedback control

**Assumption**

Each component $u_i(t, x)$ is discontinuous only on a surface

$$S_i = \{(t, x) \in \mathbb{R}^n : s_i(t, x) = 0\},$$

where functions $s_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ are smooth, i.e. $s_i \in C^1(\mathbb{R}^{n+1})$. 
\[ \dot{x}(t) = f(t, x(t), U(t, x(t))), \quad t \in \mathbb{R}, \]

where \( U(t, x) = (U_1(t, x), U_2(t, x), \ldots, U_m(t, x))^T \) and

\[
U_i(t, x) = \begin{cases} 
\{u_i(t, x)\}, & s_i(t, x) \neq 0 \\
\text{co} \left\{ \lim_{(t_j, x_j) \to (t, x)} u_i(t_j, x_j), \lim_{(t_j, x_j) \to (t, x)} u_i(t_j, x_j) \right\}, & s_i(t, x) = 0
\end{cases}
\]

The set \( f(t, x, U(t, x)) \) is \textbf{non-convex} in general case.
Example (Utkin Regularization)

\[ u(x) = - \text{sign}[x] \quad \text{and} \quad U(x) = \overline{\text{sign}}[x] \]

\[
\text{sign}[\rho] = \begin{cases} 
1 & \text{if } \rho > 0 \\
-1 & \text{if } \rho < 0 \\
0 & \text{if } \rho = 0 
\end{cases}, \quad \overline{\text{sign}}[\rho] = \begin{cases} 
1 & \text{if } \rho > 0 \\
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\end{cases}
\]
Equivalent Control (Utkin Solution)

**Definition**

An absolutely continuous function \( x : \mathcal{I} \rightarrow \mathbb{R}^n \) defined on some interval or segment \( \mathcal{I} \) is called a solution of (DisContSys) if there exists a measurable function \( u_{eq} : \mathcal{I} \rightarrow \mathbb{R}^m \) such that \( u_{eq}(t) \in U(t, x(t)) \) and

\[
\dot{x}(t) = f(t, x(t), u_{eq}(t)) \quad \text{almost everywhere on} \quad \mathcal{I}.
\]
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$$\dot{x}(t) = f(t, x(t), u_{eq}(t))$$

almost everywhere on $\mathcal{I}$.

![Filippov and Utkin definitions of equivalent control](image)

Equivalent control (Utkin 1967): $s(x) = 0$ and $\frac{\partial s(x)}{\partial x} f(t, x, u_{eq}) = 0$
Example (Equivalent Control)

\[ \begin{align*}
    \dot{x}_1 &= u \\
    \dot{x}_2 &= (2u^2 - 1)x_2 \\
    u(t) &= -\text{sign}[x_1(t)]
\end{align*} \]
Example (Equivalent Control)

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\end{align*} \]

\[ u(t) = -\text{sign}[x_1(t)] \]

Filippov definition

\[ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \in \begin{bmatrix} -\text{sign}[x_1(t)] \\ x_2(t) \end{bmatrix} \]
Example (Equivalent Control)

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Unstable
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-x_2(t)
\end{bmatrix}
\]

Stable
Example (Utkin 1970s)

\[
\begin{align*}
\dot{x}_1 &= 0.3x_2(t) + x_1(t)u(t), \\
\dot{x}_2 &= -0.7x_1(t) + 4x_1^3(t)u(t),
\end{align*}
\]

\[u(t) = -\text{sign}[x_1(t)s(t)],\]

\[s(t) = x_1(t) + x_2(t),\]
Example (Utkin 1970s)

\[ \dot{x}_1 = 0.3x_2(t) + x_1(t)u(t), \]
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Aizerman-Pyatniskii Regularization (Filippov 1988)

\[ \dot{x} \in \text{co}(f(t, x, U(t, x)), \quad t \in \mathbb{R} \]

**Definition**

An absolutely continuous function \( x : \mathcal{I} \rightarrow \mathbb{R}^n \) defined on some interval or segment \( \mathcal{I} \) is called a solution of (DiscRHS) if it satisfies the differential inclusion (DiffInc) almost everywhere on \( \mathcal{I} \).
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Presence of fast actuators (Fridman 2001, 2002)

Actuators

1st order actuator

\[ \mu \dot{z}_1 = -2z_1 - u(s), \]

2nd order actuator

\[ \begin{align*}
\mu \dot{z}_1 & = z_2, \\
\mu \dot{z}_2 & = -2z_1 - 3z_2 - u(s),
\end{align*} \]

Plant

\[ \begin{align*}
\dot{s} & = z, \\
\dot{x} & = u^4 - u^2 + \beta x, \\
u(s) & = \text{sign}[s],
\end{align*} \]
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Plant

\[ \dot{s} = z, \quad \dot{x} = u^4 - u^2 + \beta x, \quad u(s) = \text{sign}[s], \]
Reduced Order System

\[ \mu = 0 \Rightarrow z_1 = -u(s)/2, \quad \dot{s} = -u/2, \quad \dot{x} = (u^4 - u^2 + \beta)x, \]
Reduced Order System

\[ \mu = 0 \Rightarrow z_1 = -u(s)/2, \quad \dot{s} = -u/2, \quad \dot{x} = (u^4 - u^2 + \beta)x, \]

Sliding Dynamics (Filippov=Utkin)
1st order actuator \((z(t), s(t)) \to 0\) Sliding dynamics \(\dot{x} = \beta x \Rightarrow \text{Unstable}

2nd order actuator

\[\exists \left( z_0 \left( \frac{t}{\mu} \right), s_0 \left( \frac{t}{\mu} \right) \right) - \text{Periodic Solution} \]

\[\exists \bar{\beta}(\mu) : \forall \beta < \bar{\beta}(\mu) \exists \gamma : \]

\[-\gamma = \int_{0}^{T} \left[ (2z_1(\tau))^4 - (2z_2(\tau))^2 \right] d\tau \]

\[\Rightarrow \dot{x} = -(\gamma - \beta)x \]

Could be stable
Theorem (Utkin 1992, Zolezzi 2002)

Let a right-hand side of the system (DiscRHS) be affine with respect to control:

\[ f(t, x, u(t, x)) = a(t, x) + b(t, x)u(t, x), \]

where \( a : \mathbb{R}^{n+1} \to \mathbb{R}^n \), \( b : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times m} \), \( a, b \in \mathbb{C} \) and \( u : \mathbb{R}^{n+1} \to \mathbb{R}^m \) is a piecewise continuous function \( u(t, x) = (u_1(t, x), \ldots, u_m(t, x))^T \), such that \( u_i \) has a unique switching surface \( s_i(x) = 0 \), \( s_i \in C^1(\mathbb{R}^n) \).
Equivalence of Definitions

**Theorem (Utkin 1992, Zolezzi 2002)**

Let a right-hand side of the system (DiscRHS) be affine with respect to control:

\[ f(t, x, u(t, x)) = a(t, x) + b(t, x)u(t, x), \]

where \( a : \mathbb{R}^{n+1} \to \mathbb{R}^n, b : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times m}, a, b \in \mathbb{C} \) and \( u : \mathbb{R}^{n+1} \to \mathbb{R}^m \) is a piecewise continuous function \( u(t, x) = (u_1(t, x), \ldots, u_m(t, x))^T \), such that \( u_i \) has a unique switching surface \( s_i(x) = 0 \), \( s_i \in \mathbb{C}^1(\mathbb{R}^n) \).

Definitions of Filippov, Utkin and Aizerman-Pyatnitskii are equivalent iff

\[ \det \left( \nabla^T s(x)b(t, x) \right) \neq 0 \text{ if } (t, x) \in S, \]

where \( s(x) = (s_1(x), s_2(x), \ldots, s_m(x))^T \), \( \nabla s(x) \in \mathbb{R}^{n \times m} \) is the matrix of partial derivatives \( \frac{\partial s_i}{\partial x_i} \) and \( S \) is a discontinuity set of \( u(t, x) \).
\[ \dot{x} = Ax(t) + cu_1(t) + bu_2(t), \]
\[ t > 0, \quad x(\cdot) = (x_1(\cdot), x_2(\cdot))^T \in \mathbb{R}^2, \]
\[ A \in \mathbb{R}^{2 \times 2}, \quad b = (0, 1)^T, \]
\[ u_1(t) = -\text{sign}[x_1(t)], \]
\[ u_2(t) = -\text{sign}[x_1(t)], \]
\[ c = (1, 0)^T, \]
Example (Neimark 1961)

\[ \dot{x} = Ax(t) + cu_1(t) + bu_2(t), \]
\[ t > 0, \quad x(\cdot) = (x_1(\cdot), x_2(\cdot))^T \in \mathbb{R}^2, \]
\[ A \in \mathbb{R}^{2 \times 2}, \quad b = (0, 1)^T, \]
\[ c = (1, 0)^T, \]

Filippov definition

\[ \dot{x} \in \{ Ax \} \dot{+} (b + c) \cdot \text{sign}[x_1] \]

Utkin definition

\[ \dot{x} \in \{ Ax \} \dot{+} b \cdot \text{sign}[x_1] + c \cdot \text{sign}[x_1]. \]
Models of sliding mode control systems usually have the form

\[ \dot{x}(t) = f(t, x(t), u(t, x(t)), d(t)), \quad t \in \mathbb{R}, \]

- \( x(\cdot) \in \mathbb{R}^n \) is the vector of system states,
- \( u(\cdot, \cdot) \in \mathbb{R}^m \) is the vector of control inputs,
- \( d(\cdot) \in R^k \) is the vector of disturbances,
- the function \( f : \mathbb{R}^{n+m+k+1} \rightarrow \mathbb{R}^n \) is assumed to be continuous,
- the control function \( u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m \) is piecewise continuous,
- the vector-valued function \( d : R \rightarrow \mathbb{R}^k \) is assumed to be locally measurable and bounded as follows:

\[ d_i^{\min} \leq d_i(t) \leq d_i^{\max} \]

where \( d(t) = (d_1(t), d_2(t), \ldots, d_k(t))^T, \ t \in \mathbb{R} \).
Example (Disturbed sliding mode system)

Consider the simplest disturbed sliding mode system

\[
\dot{x}(t) = -d_1(t) \text{sign}[x(t)] + d_2(t),
\]  

(Ex1)

where \( x \in \mathbb{R} \), unknown functions \( d_i : \mathbb{R} \to \mathbb{R} \) are bounded by

\[
d_i^{\text{min}} \leq d_i(t) \leq d_i^{\text{max}}, \quad i = 1, 2.
\]
Example (Disturbed sliding mode system)

Consider the simplest disturbed sliding mode system

$$\dot{x}(t) = -d_1(t) \text{sign}[x(t)] + d_2(t),$$  \hspace{1cm} (Ex1)

where $x \in \mathbb{R}$, unknown functions $d_i : \mathbb{R} \rightarrow \mathbb{R}$ are bounded by

$$d_i^{\text{min}} \leq d_i(t) \leq d_i^{\text{max}}, \hspace{0.5cm} i = 1, 2.$$  

Obviously, all solutions of the system (Ex1) belong to a solution set of the following extended differential inclusion

$$\dot{x}(t) \in -[d_1^{\text{min}}, d_1^{\text{max}}] \cdot \text{sign}[x(t)] + [d_2^{\text{min}}, d_2^{\text{max}}].$$  \hspace{1cm} (Ex2)

Stability of the system (Ex2) implies the same property for (Ex1). In particular, for $d_1^{\text{min}} > \max\{|d_2^{\text{min}}|, |d_2^{\text{max}}|\}$ both these systems have asymptotically stable origin.
Extended Differential Inclusion

All further considerations deal with the extended differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad t \in \mathbb{R},$$

where

$$F(t, x) = \text{co}\{f(t, x, U(t, x), D)\},$$

the set-valued function $U(t, x)$ is defined by (ValFunc) and

$$D = \begin{pmatrix}
[d_{1\min}, d_{1\max} \\
[d_{2\min}, d_{2\max}] \\
\vdots \\
[d_{k\min}, d_{k\max}]
\end{pmatrix}$$
Outline

1. Preliminaries
   - Absolute Continuity
   - Upper semi-continuity

2. Equations with Discontinuous RHS
   - Historical Remarks

3. Regularization Procedure for ODE with Discontinuous RHS
   - Filippov Solutions
   - Utkin Solutions
   - Aizerman-Pyatniskii Solutions

4. Disturbed Systems and Extended Differential Inclusions
   - Disturbances and Differential Inclusions

5. Existence of Solutions
   - Existence Conditions
Local existence conditions

Theorem (Filippov 1960)

Let

- \( F : G \rightarrow 2^{\mathbb{R}^n} \) be upper semi-continuous at each point of the set
  \[
  G = \{(t, x) \in \mathbb{R}^{n+1} : |t - t_0| \leq a \text{ and } \|x - x_0\| < b,\]

  where \( a, b \in \mathbb{R}_+, \ t_0 \in \mathbb{R}, \ x_0 \in \mathbb{R}^n; \)

- \( F(t, x) \) be nonempty, compact and convex for \( (t, x) \in G; \)

- there exists \( K > 0 \) such that \( \rho(0, F(t, x)) < K \) for \( (t, x) \in G; \)
Local existence conditions

Theorem (Filippov 1960)

Let

- $F : G \to 2^{\mathbb{R}^n}$ be upper semi-continuous at each point of the set $G = \{(t, x) \in \mathbb{R}^{n+1} : |t - t_0| \leq a \text{ and } \|x - x_0\| < b\}$,

  where $a, b \in \mathbb{R}_+, t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n$;
- $F(t, x)$ be nonempty, compact and convex for $(t, x) \in G$;
- there exists $K > 0$ such that $\rho(0, F(t, x)) < K$ for $(t, x) \in G$;

then $\exists x : \mathbb{R} \to \mathbb{R}^n$ - absolutely continuous and defined at least on $[t_0 - \alpha, t_0 + \alpha]$, $\alpha = \min\{a, b/K\}$, such that $x(t_0) = x_0$ and the inclusion

$$\dot{x}(t) \in F(t, x(t))$$

holds almost everywhere on $[t_0 - \alpha, t_0 + \alpha]$. 

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Equations with Discontinuous RHS
Lemma (Filippov 1959)

Let

- a function \( f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n \) be continuous;
- a set-valued function \( U : \mathbb{R}^{n+1} \to 2^{\mathbb{R}^m} \) be defined and upper-semicontinuous on an open set \( \mathcal{I} \times \Omega \), where \( \Omega \subseteq \mathbb{R}^n \);
- \( U(t, x) \) be nonempty, compact and convex for every \( (t, x) \in \mathcal{I} \times \Omega \).
- a function \( x : \mathbb{R} \to \mathbb{R}^n \) be absolutely continuous on \( \mathcal{I} \), \( x(t) \in \Omega \) for \( t \in \mathcal{I} \),
- \( \dot{x}(t) \in f(t, x(t), U(t, x(t))) \) almost everywhere on \( \mathcal{I} \);
Lemma (Filippov 1959)

Let

- a function \( f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n \) be continuous;
- a set-valued function \( U : \mathbb{R}^{n+1} \to 2^{\mathbb{R}^m} \) be defined and upper-semicontinuous on an open set \( I \times \Omega \), where \( \Omega \subseteq \mathbb{R}^n \);
- \( U(t, x) \) be nonempty, compact and convex for every \((t, x) \in I \times \Omega\);
- a function \( x : \mathbb{R} \to \mathbb{R}^n \) be absolutely continuous on \( I \), \( x(t) \in \Omega \) for \( t \in I \),
- \( \dot{x}(t) \in f(t, x(t), U(t, x(t))) \) almost everywhere on \( I \);

Then there exists a measurable function \( u_{eq} : R \to \mathbb{R}^m \) such that

\[
u_{eq}(t) \in U(t, x(t)) \quad \text{and} \quad \dot{x}(t) = f(t, x(t), u_{eq}(t)) \]

almost everywhere on \( I \).
Non-local existence conditions

Theorem (Gelig et al. 1978)

Let a set-valued function $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be defined and upper-semicontinuous in $\mathbb{R}^{n+1}$.
Let $F(t, x)$ be nonempty, compact and convex for any $(t, x) \in \mathbb{R}^{n+1}$.
If there exists a real valued function $L : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ such that

$$
\rho(0, F(t, x)) \leq L(\|x\|) \quad \text{and} \quad \int_0^{+\infty} \frac{1}{L(r)} dr = +\infty,
$$

then for any $(t_0, x_0) \in \mathbb{R}^{n+1}$ the system (DiffInc) has a solution $x(t) : x(t_0) = x_0$ defined for all $t \in \mathbb{R}$. 

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Equations with Discontinuous RHS
Summary

- Stability property of ODE with discontinuous RHS depends on definition of a solution.
- All introduced definitions may be equivalent in the case of affine control systems with discontinuous input.
- Analysis of the disturbed systems can be reduced to differential inclusions.