Continuous Nested Algorithms : The Fifth Generation of Sliding Mode Controllers

Leonid Fridman[1], Jaime A. Moreno[2], Bijnan Bandyopadhyay[3], Shyam Kamal[3], and Asif Chalanga[3]

[1] Facultad de Ingeniería Universidad Nacional Autónoma de México (UNAM)
Coyoacán D.F. 04510, Mexico
lfridman@unam.mx
[2] Instituto de Ingeniería Universidad Nacional Autónoma de México (UNAM)
Coyoacán. 04510, Mexico
JMorenoP@ii.unam.mx
[3] Systems and Control Engineering, Indian Institute of Technology Bombay, Mumbai-400 076, India
{bijnan,shyam,asif}@sc.iitb.ac.in

Abstract. The history and evolution of Sliding Mode Controllers in the last three decades is revisited. The new generation of continuous sliding-mode controllers, and continuous nested sliding-mode controllers is presented. Such controllers generate an continuous control signal, ensuring, for the systems with relative degree r, the finite–time convergence to the $(r+1)-th$ sliding-mode set using only information on the sliding output and its derivatives up to the $(r-1)$ order.

In this book it is natural to recall the past and to think about the future. This chapter is an attempt to give a viewpoint on the stages of development of the Sliding-Mode Control(SMC) theory in the last decades. We will show that each decade the SMC community has been able to generate families of controllers with much better properties than before, and propose arbitrary-order continuous SMC algorithms which can significatively reduce the chattering and improve the precision.

1 The First Generation of Sliding Modes Controllers

The classical theory of first order SMC was established by 1980 and later reported in Prof. Utkin's monograph in Russian, in 1981 (English version [35]). In his monograph Porfessor Utkin clearly stated the two-step procedure for sliding-mode control design:

1. Sliding surface design;
2. Discontinuous (relay or unit) controllers ensuring the sliding modes.

The main advantages of the first order SMC are the following:

- theoretically exact compensation (insensitivity) w.r.t. bounded matched uncertainties [7];

X. Yu and M. Önder Efe (eds.), *Recent Advances in Sliding Modes*,
Studies in Systems, Decision and Control 24, DOI: 10.1007/978-3-319-18290-2_2

- reduced order of sliding equations;
- finite–time convergence to the sliding surface.

However, the following disadvantages were evident:

- chattering;
- the sliding variables converge in finite–time but the state variables only converge asymptotically;
- the sliding surface design is restricted to have relative degree one with respect to the control, i.e., higher order derivatives are required for the sliding surface design.

2 The Second Generation of SMC: Second Order Sliding Modes

By the early 80's, the control community had understood that the main disadvantage of SMC is the "chattering" effect [35],[36]. It has been shown that this effect is mainly caused by unmodelled cascade dynamics which increase the system's relative degree, and perturb the ideal sliding mode [3],[14],[36], i.e. in order to adjust the chattering it is necessary that not only the sliding variable tends to zero, but also its derivative.

2.1 Second Order Sliding Modes

The second order sliding modes (SOSM) concept was introduced in the Ph.D. dissertation of A. Levant (Levantovskii).

Consider a second order uncertain system

$$\ddot{\sigma} = f\left(\sigma, \dot{\sigma}, t\right) + g\left(\sigma, \dot{\sigma}, t\right)\nu,$$

where σ and $\dot{\sigma}$ are the system state, $\sigma = x_1$ is the system output, $\nu \in R$ is the scalar control and $f\left(\sigma, \dot{\sigma}, t\right)$ represents unknown uncertainties/perturbations. It is also assumed that all the partial derivatives of $f(\sigma, \dot{\sigma}, t)$ are bounded on compacts and $g\left(\sigma, \dot{\sigma}, t\right) \neq 0$ is known. Then, one can write

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = f\left(x, t\right) + g\left(x, t\right)\nu, \end{cases} \tag{1}$$

where $x_2 = \dot{\sigma}$ and $x = [x_1, x_2]^T$. For simplicity, it will be assumed that $g(x, t) > 0$ for all t, x. Defining $\nu = g^{-1}(x, t)u$, system (2.1) can be written as

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = u + f(x, t). \end{cases} \tag{2}$$

The main objective of SOSM was to design a control u such that the origin of system (2) is finite-time stable, in spite of the uncertainties/perturbations $f(x, t)$, with $|f(x, t)| < f^+$ for all t, x. For the above mentioned goal, a controller is proposed in the next section.

Here, and always below, the solution of the all systems will be understood in the sense of A. Filippov[12].

2.2 Twisting Algorithm

The first and simplest SOSM algorithm is the so-called "Twisting Algorithm"(TA)[10]. For a relative degree two system the TA takes the form

$$u = -a\,\mathrm{sign}(x_2) - b\,\mathrm{sign}(x_1),\ \ b > a + f^+, a > f^+.$$

Under the assumption of known bounds for f^+, and with parameters a and b of the controller chosen appropriately [10], the twisting algorithm ensures finite-time exact convergence of both x_1 and x_2, i.e. there exists $T > 0$ such that, for all $t > T$, $x_1(t) = x_2(t) = 0$. Thus, the TA is said to be a SOSM control algorithm since it provides a (stable) "second order sliding mode" at the origin. An example trajectory can be seen in the Figure 1.

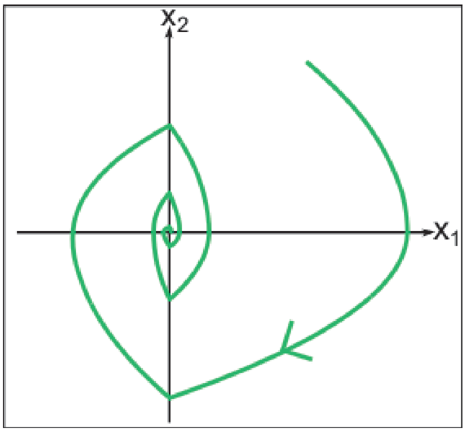


Fig. 1. Example trajectory of the Twisting algorithm

2.3 Terminal Algorithm and Singularity of Switching Surface

Consider the second order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u(x), \tag{3}$$

where the terminal sliding mode control input u is given by [27],[39].

$$u(x) = -\alpha\,\mathrm{sign}(s(x)), \quad s(x) = x_2 + \beta\sqrt{|x_1|}\,\mathrm{sign}(x_1). \tag{4}$$

By taking the time derivative of the switching surface, it is obtained

$$\dot{s}(x) = \dot{x}_2 + \beta\frac{x_2}{2\sqrt{|x_1|}} = -\alpha\,\mathrm{sign}(s(x)) + \beta\frac{x_2}{2\sqrt{|x_1|}}. \tag{5}$$

This means that the derivative of the switching surface $s(x)$ is singular for $x_1 = 0$, and, consequently, **the relative degree of the switching surface does not**

exist. From now on, we will call the switching surface $s(x)$, singular. On the switching surface $x_2 = -\beta\sqrt{|x_1|}\,\mathrm{sign}(x_1)$, it occurs that

$$\dot{s} = -\alpha\,\mathrm{sign}(s(x)) - \frac{\beta^2}{2}\,\mathrm{sign}(x_1).$$

It is clear that under condition $\beta^2 < 2\alpha$, the sliding on the surface $s(x) = 0$ exists, and two types of behavior for the solution of the system are possible [24], [33], [34]:

Terminal Mode. For the case when $\beta^2 < 2\alpha$, the trajectories of the system reach the surface $s(x) = 0$ and remain there for all the future time. This kind of behavior can be seen in Figure 2.

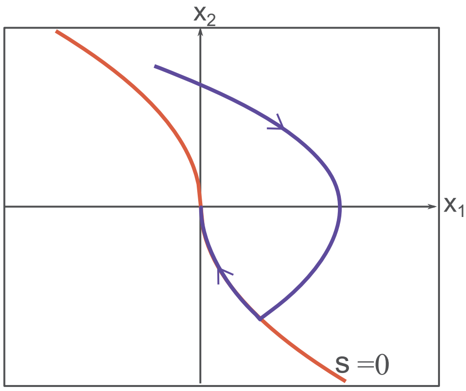


Fig. 2. Terminal mode with $\beta = 3,\ \alpha = 5$

The ideal sliding and computational chattering start when the solution reaches $s(x) = x_2 + \beta\sqrt{|x_1|}\,\mathrm{sign}(x_1)$.

Twisting Mode. When the controller parameters are chosen such that $\beta^2 > 2\alpha$, the trajectories of the system do not slide on the surface $s(x) = 0$. This behavior is exemplified in Figure 3. Note that the computational chattering does not start until the states reach the system's origin.

As it has been seen from (5), there is an issue of singularity of the switching surface. Such issue has been overcame by rewriting the function s as follows [11]

$$\bar{s}(x) = \beta^2 x_1 + x_2^2\,\mathrm{sign}(x_2).$$

Note that $s(x) = 0$, and $\bar{s}(x) = 0$ describe the same switching surface.

Precision of SOSM. The main advantage of SOSM is that they are homogeneous, with weights {2,1} ([1],[30]). As it is shown in [19], the order of precision, determined by the weights of homogeneity in terms of the discretization step δ,

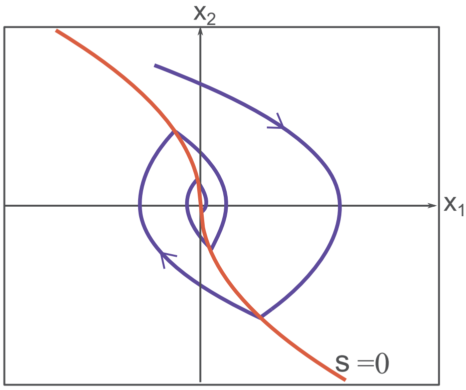


Fig. 3. Twisting mode with $\beta = 4,\ \alpha = 5$

is $O(\delta^2)$ with respect to the sliding output and $O(\delta)$ with respect to its derivative. Moreover, in the presence of fast actuator dynamics with time constant μ, the precision order is $O(\mu^2)$ with respect to the sliding output and $O(\mu)$ with respect to its derivative [5],[26].

2.4 Discussion about SOSM

Advantages of SOSM

1. **SOSM ensures the quadratic precision of convergence with respect to the sliding output.**
2. For one degree of freedom mechanical systems, both the Twisting and the Terminal controllers provide dynamic collapse, i.e. the sliding surface design is no longer needed.
3. For systems with relative degree r, the order of the sliding dynamics is reduced up to $(r-2)$. The design of the sliding surface of order $(r-2)$ is still necessary.

However, the following problems remain open:

- **SOSM algorithms for systems with relative degree two still produce a discontinuous control signal, i.e., they can not reduce the chattering substantially.**
- The problem of exact finite-time stabilization (dynamic collapse) and exact disturbance compensation for SISO systems with arbitrary relative degree still persists.

Chattering Attenuation Strategy Based on SOSM. Under additional assumptions regarding the smoothness of the system, the SOSM controllers(see also [2]) have been used to attenuate chattering in systems with relative degree one, by including an integrator in the control input. Consider the following system

$$\dot{X} = F(t,X) + G(t,X)u, X \in R^n, u \in R$$
$$\dot{u} = v,$$

where F is a function with known upper bound. The relative degree one switching variable $\sigma(X)$ is designed such that it satisfies the equation $\dot{\sigma} = f(x,t)+g(x,t)u$. Then, defining for example a Twisting-like control $v = -a\,\mathrm{sign}(\dot{\sigma}(X)) - b\,\mathrm{sign}(\sigma(X))$, or a Terminal-like control $v = -\alpha\,\mathrm{sign}(s(\sigma(X)))$, and selecting appropriate parameters, we will have an continuous control signal u, ensuring finite-time convergence to the surface $\sigma(X) = 0$.

2.4.1 The First Criticism of SOSM

In the end of the 80's, the SOSM were strongly criticized. The main point of this criticism is the that anti-chattering strategy for a first order sliding mode uses the derivative $\dot{\sigma}$. Thus, if by any reason it is possible to measure $\dot{\sigma} = f(t,\sigma)+g(t,\sigma)u$ and, additionally, $g(t,\sigma)$ is also known, then the uncertainty $f(t,\sigma) = \dot{\sigma}-g(t,\sigma)u$ is also known and can be compensated without any discontinuous control! In this case, what is the reason for theq use of a SMC?

In the late eighties it was clear that, in order to adjust the chattering for a relative degree one sliding variable, an continuous control signal should be generated without requiring information on the derivative of the sliding variable, i.e. on the perturbations.

3 Third Generation of Sliding Modes Controllers: The Super-Twisting Algorithm

The Super–Twisting Algorithm (STA)[19]:

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)u, \\ u &= -k_1|x|^{\frac{1}{2}}\,\mathrm{sign}(x) + v, \\ \dot{v} &= -k_2\,\mathrm{sign}(x), \end{aligned} \tag{6}$$

where f is any Lipschitz bounded uncertainty/disturbance, for some constants k_1 and k_2, ensures [19] exact finite time convergence to the second sliding-mode set $x(t) = \dot{x}(t) = 0$, $\forall t \geq T$ without usage of $\dot{x}$. If we consider system (6) having x as the measured output, the STA is an output-feedback controller for a system of one dimension.

3.0.2 Robust Exact Differentiator

This last property of the STA allowed to construct the "robust exact" sliding-mode differentiator [20] and gave further impetus to the development of the mathematical theory and applications of SOSM algorithms. We now briefly describe the idea behind it. Let $f(t)$ be a signal to be differentiated and assume that $|\ddot{f}(t)| \leq L$, with L being a known constant. Take $x_1 = f, x_2 = \dot{f}$; then the problem can be reformulated as finding an observer for

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \ddot{f}, \quad y = x_1,$$

where $\ddot{f}(t)$ is considered as a bounded perturbation. Since the STA does not require derivatives, which in this case would be the state x_2, it only uses output injection and results particularly useful in the form of a STA observer

$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1|\hat{x}_1 - y|^{\frac{1}{2}} \operatorname{sign}(\hat{x}_1 - y) + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -k_2 \operatorname{sign}(\hat{x}_1 - y)\,. \end{aligned}$$

Once the constants k_1 and k_2 are appropriately chosen, the convergence of the STA ensures that the equalities $(f - \hat{x}_1) = (\dot{f} - \hat{x}_2) = 0$ are established after a finite-time transient. Thus $\hat{x}_2$ is an estimate for the derivative $\dot{f}(t)$ and turns out to be the best possible one ([20]) in the sense of [18] when (bounded Lebesgue-measurable) noise or discretization are present. However, the difficult geometrical proof of the STA convergence remained as the main disadvantage for this algorithm, thereby preventing further generalizations.

3.0.3 Recapitulations

The use of the Super Twisting Algorithm for Lipschitz systems allows substituting a discontinuous control by means of an continuous one. Additionally, their use offers:

1. Chattering attenuation (but not its complete removal![4]).
2. Differentiator obtained using the STA:
   - finite-time exact estimation of derivatives in the absence of both noise and sampling;
   - the best possible approximation in the sense of [18] of order $O(\delta)$ w.r.t. discrete sampling and of order $O(\sqrt{\varepsilon})$ w.r.t. deterministic Lebesgue-measurable noise bounded by ε.

However, there are some disadvantages:

1. For systems with relative degree $r = 2$, the design of a sliding surface is still needed. Hence, there is finite-time convergence to the surface, but the convergence of the states to the origin is asymptotic. Moreover, in this case, the usage of STA based differentiator for the sliding surface design is not enough [6] because the reconstructed switching surface should have at least Lipschitz derivative.
2. The first order sliding mode controllers with constant gains could compensate Lebesgue but bounded perturbations. The STA is insensible to perturbations whose time derivative is bounded. However, these perturbations could grow no more fast than linear function of time, i.e., they do not need to be bounded.

4 Fourth Generation of Sliding Mode Controllers: Arbitrary Order Sliding Mode Controllers

Consider the uncertain dynamical system:

$$\begin{aligned}\dot{X} &= F(t, X) + G(t, X)u, X \in R^n, u \in R\\ \sigma &= \sigma(X, t), \in R.\end{aligned}$$

Let the output σ have a fixed and known relative degree r. In such a case, the control problem is translated into the finite-time stabilization of an uncertain differential equation or, equivalently, of the following differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u, \tag{7}$$

where C, K_m and K_M are known constants parameterizing the uncertainty of the original system.

4.1 Nested Arbitrary Order Sliding-Mode Controllers

In 2001, the first arbitrary order SM controller was introduced [21], combining relay controller with hierarchical terminal sliding modes [38]. Such controllers solve the finite-time exact stabilization problem for an output with an arbitrary relative degree, in the presence of bounded Lebesgue measurable uncertainties.

Given the relative degree r of the output, "Nested" higher order sliding-mode(HOSM) controllers are constructed using a recursion, generalizing the singular Terminal Algorithm. The following is the recursion for the Singular Terminal Algorithm. Let p be the least common multiple of $1, 2, \ldots, r$. Also let

$$u = -\alpha \operatorname{sign}\left(\varphi_{r-1,r}(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})\right), \tag{8}$$

where $\varphi_{0,r} = \sigma,\ N_{1,r} = |\sigma|^{\frac{r-1}{r}}$ and

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r} \operatorname{sign}(\varphi_{i-1,r}),\ N_{i,r} = \left(|\sigma|^{\frac{p}{r}} + \cdots + |\sigma^{(i-1)}|^{\frac{p}{r-1+1}}\right)^{\frac{r-i}{p}}.$$

The parameters β_i can be selected in advance in such a way that only the gain of the controller α has to be selected large enough. The algorithm provides for the finite-time stabilization of $\sigma = 0$ and, therefore, of its successive derivatives up to $r-1$. Thus, it provides for the existence of an r-th order sliding mode in the set $S_r = \{\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0\}$. In Figure 4 it is exemplified the trajectories and states for the Nested controller with $r = 3$. Since controller (8) uses the output and its successive derivatives, the HOSM arbitrary order differentiator, introduced in [23], was instrumental for the applicability of HOSM controllers. Let $\sigma(t)$ be a signal to be differentiated $k-1$ times and assume that $|\sigma^{(k)}| \leq L$,

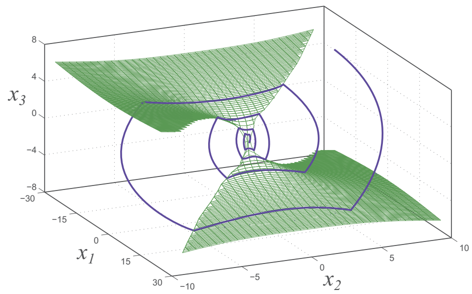


Fig. 4. System trajectory of Nested algorithm for $r = 3$

with L being a known constant. Then, the $(k-1)$-th order HOSM differentiator takes the following form

$$\begin{aligned}
\dot{z}_0 &= v_0 = -\lambda_k L^{\frac{1}{k+1}} |z_0 - \sigma|^{\frac{k}{k+1}} \operatorname{sign}(z_0 - \sigma) + z_1, \\
\dot{z}_1 &= v_1 = -\lambda_{k-1} L^{\frac{1}{k}} |z_1 - v_0|^{\frac{k-1}{k}} \operatorname{sign}(z_1 - v_0) + z_2, \\
&\vdots \\
\dot{z}_{k-1} &= v_{k-1} = -\lambda_1 L^{\frac{1}{2}} |z_{k-1} - v_{k-2}|^{\frac{1}{2}} \operatorname{sign}(z_{k-1} - v_{k-2}) + z_k \\
\dot{z}_k &= -\lambda_0 L \operatorname{sign}(z_k - v_{k-1})
\end{aligned} \tag{9}$$

where z_i is the estimation of the true derivative $\sigma^{(i)}(t)$. The differentiator ensures the finite-time exact differentiation under ideal conditions of exact measurement in continuous time. The only information needed is an upper bound, L, for $|\sigma^{(k+1)}|$. Then a parametric sequence $\{\lambda_i\} > 0$, $i = 0, 1, \ldots, k$, is recursively built, which provides for the convergence of the differentiators for each order k. In particular, the parameters $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ are enough up until the 5-th differentiation order. With discrete sampling, the differential equations are replaced by their Euler approximations. This differentiator provides for the best possible asymptotic accuracy in the presence of input noises or discrete sampling [19,18] for the rth derivative:

- order $O(\delta)$ with respect to discrete sampling,
- order $O(\varepsilon^{\frac{1}{r+1}})$ with respect to bounded deterministic Lebesgue measurable noise.

The use of the HOSM arbitrary order differentiator together with the HOSM arbitrary order controller allowed the design and the implementation of a nested arbitrary-order HOSM output-feedback controller for uncertain single-input single-output (SISO) systems ensuring the finite-time output stabilization in spite of disturbances. The block diagram for implementation of the output-feedback nested HOSM controller is presented in Figure 5.

4.1.1 Discussion about Nested HOSM

Nested HOSM algorithm ensures exact finite-time stabilization (dynamic collapse) of the output σ and exact disturbance compensation for SISO systems with relative degree r, using information on $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$.

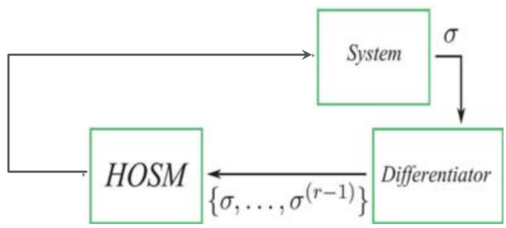


Fig. 5. The implementation of the output-feedback nested HOSM controller

Some of its advantages are:

- **SOSM ensure the r-th order precision for the sliding output with respect to the discretization step and fast parasitic dynamics [22],[26].**
  item The sliding surface design is no longer needed.

However, the nested HOSM algorithms for relative degree r systems still produces a discontinuous control signal, i.e., they can not reduce the chattering substantially.

5 Fifth Generation of SMC: Continuous Arbitrary Order Sliding-Mode Controllers

In this section we propose an arbitrary order Continuous Nested Sliding Mode Algorithm(CNSMA). The CNSMA provides, for relative degree r systems with respect to the output,

- continuous control signal;
- finite-time convergence to the $(r+1)$-th order sliding-mode set;
- derivatives of the output up to the $(r-1)$ order.

Firstly for the systems with relative degree two we will introduce two versions of the Continuous Terminal Sliding Mode Algorithm(CTSMA), as a combination of Super-Twisting with both versions of Terminal Algorithm: singular and nonsingular. It will be shown that the CTSMA has also the above mentioned properties of the CNSMA for the systems with relative degree two. The possibilities to prove their convergence will be discussed.

Than the CNSMA for the systems with arbitrary relative degree is suggested.

In this section the following notation is used, for a real variable $z \in \mathbb{R}$ to a real power $p \in \mathbb{R}$, $\lfloor z \rceil^p = |z|^p sgn(z)$, therefore $\lfloor z \rceil^2 = |z|^2 sgn(z) \neq z^2$. If p is an odd number, this notation does not change the meaning of the equation, i.e. $\lfloor z \rceil^p = z^p$. Therefore

$$\begin{aligned} &\lfloor z \rceil^0 = sgn(z), \quad \lfloor z \rceil^0 z^p = |z|^p, \quad \lfloor z \rceil^0 |z|^p = \lfloor z \rceil^p \\ &\lfloor z \rceil^p \lfloor z \rceil^q = |z|^p sgn(z) |z|^q sgn(z) = |z|^{p+q} \end{aligned} \tag{10}$$

5.1 Continuous Terminal Sliding Mode Algorithm

Continuous terminal sliding-mode algorithms are defined in the following way:

(a) Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA);
(b) Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA).

Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA)

Suppose that the control input u is defined as

$$u = -\, k_1 \lfloor \phi \rceil^{1/2} - k_3 \int_0^t \lfloor \phi \rceil^0 d\tau, \tag{11}$$

or

$$u = -k_1 \lfloor \phi \rceil^{1/2} + L, \quad \dot{L} = -k_3 \lfloor \phi \rceil^0, \tag{12}$$

where $\phi = \left(x_2 + k_2 \lfloor x_1 \rceil^{2/3}\right)$, and $k_1, k_2,\, k_3$ are appropriate positive gains. Substituting the control (12) into (2), the closed loop system becomes

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lfloor \phi \rceil^{1/2} + L + f(x,t) \\ \dot{L} &= -k_3 \lfloor \phi \rceil^0. \end{cases} \tag{13}$$

Suppose $x_3 = L + f(x,t)$, then one can rewrite (13) as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lfloor \phi \rceil^{1/2} + x_3 \\ \dot{x}_3 &= -k_3 \lfloor \phi \rceil^0 + \rho, \end{cases} \tag{14}$$

where $\rho = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}$, and it is assumed that it satisfies $|\rho| \leq \Delta$.

Proposed algorithm (14) can be interpreted as a combination of the Super-Twisting algorithm with the Singular Terminal Sliding mode.

5.2 Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA)

Suppose that the control input u is defined as

$$u = -\, k_1 \lfloor \phi_N \rceil^{1/3} - k_3 \int_0^t \lfloor \phi_N \rceil^0 d\tau, \tag{15}$$

or

$$u = -k_1 \lfloor \phi_N \rceil^{1/3} + L, \quad \dot{L} = -k_3 \lfloor \phi_N \rceil^0, \tag{16}$$

where, $\phi_N = \left(x_1 + k_2\lfloor x_2\rceil^{3/2}\right)$ and $k_1, k_2,\ k_3$ are appropriate positive gains. Substituting the control (16) into (2), the closed loop system becomes

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1\lfloor \phi_N\rceil^{1/3} + L + f(x,t) \\ \dot{L} &= -k_3\lfloor \phi_N\rceil^{0}. \end{cases} \tag{17}$$

Suppose $x_3 = L + f(x,t)$, then one can rewrite (17) as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1\lfloor \phi_N\rceil^{1/3} + x_3 \\ \dot{x}_3 &= -k_3\lfloor \phi_N\rceil^{0} + \rho, \end{cases} \tag{18}$$

where $\rho = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}$ and assume that it satisfy $|\rho| \leq \Delta$.

Proposed algorithm (18) can be viewed as a combination of the Super-Twisting algorithm with the Nonsingular Terminal Sliding Mode algorithm.

5.2.1 Discussion about the CSTSMA and CNSTMA

Continuous singular/nonsingular terminal sliding-mode algorithms (14) and (18) are homogeneous of degree $\delta_f = -1$, with weights $\varrho = \{3, 2, 1\}$. The main advantage of this algorithm is that, the only information needed, for the finite time convergence of all three variables x_1, x_2 and x_3, is the output (x_1) and its derivative (x_2). It is also obvious that $\dot{x}_2 = 0$ because ϕ, which is a function of x_1, x_2 and x_3, equals to zero. The precision of the output tracking $\sigma, \dot{\sigma}$ and $\ddot{\sigma}$, corresponding to a 3^{rd} order sliding mode.

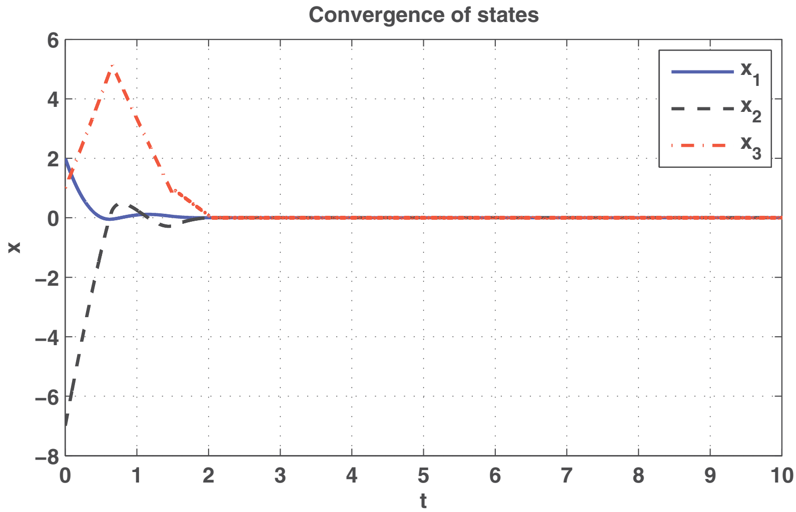
The parameters used in the simulation of were

- initial conditions $x_1(0) = 2$ and $x_2(0) = -7$
- gains $k_1 = 6, k_2 = 5$ and $k_3 = 6$

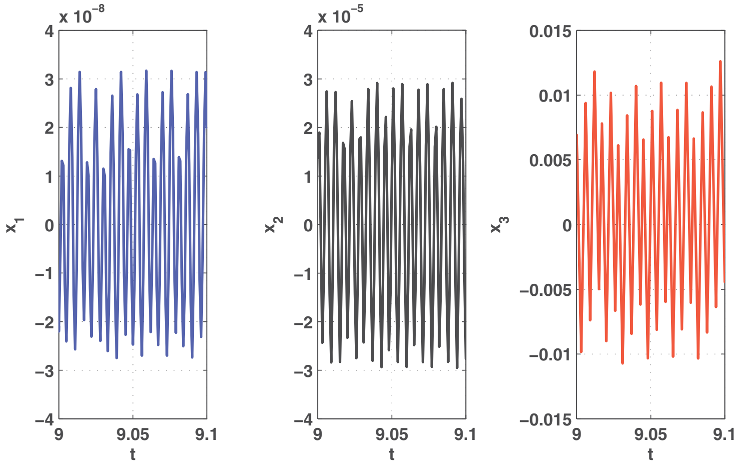
After substituting the control u in (2), the closed loop system is the same as in (14). Figure 6 shows that the convergence and precision of the states x_1, x_2 and x_3 are $10^{-9}, 10^{-6}$ and 10^{-3} respectively, when the simulation step of the Euler algorithm is set to $\tau = 10^{-3}$. It evident from the simulation that the precision corresponds to a third order sliding mode.

Figure (7) shows the convergence of the states, the phase portrait, the control input and the perturbation estimation of a second order uncertain plant with 3-CSTSMC as a controller. It is noticeable from the phase portrait in Fig. (7)(b) that switching surface $\phi = 0$ does not seem to be a sliding surface, and shows a behavior typical of the second order sliding mode known as Twisting controller.

The time evolution of the states of system (18) with u as a Continuous Nonsingular Terminal Sliding Mode Control(CNTSMC) are given in Figure 9, where the value of perturbation is again taken as $f = 2 + 4\sin(t/2) + 0.6\sin(10t)$ and gains are selected as $k_1 = 13.4,\ k_2 = 3.3,\ k_3 = 25$. It is clear from the figure that all the states converge to zero, despite of the perturbation f. Figure 10 shows the precision of each of the states when the simulation step of the Euler



(a)



(b)

Fig. 6. Convergence and precision of states with $\tau = 0.001$ for 3-CSTSMA

algorithm is set to $\tau = 10^{-3}$ (Fig. 10a), or $\tau = 10^{-4}$ (Fig. 10b). From them we can calculate the (precision) coefficients: $\nu_3 = 80$, $\nu_2 = 80$ and $\nu_1 = 1200$. They show that the precision corresponds to a third-order sliding mode.

In Figure 11 the phase portrait of the plant's states x_1 and x_2 is shown, along with the switching curve $\phi_N = 0$. It is noticeable that the trajectory reaches the switching surface and then slides along it, until the origin is reached in finite time. This is also clear from the behavior of $\phi = \phi_N$ in Figure 9, that also appears

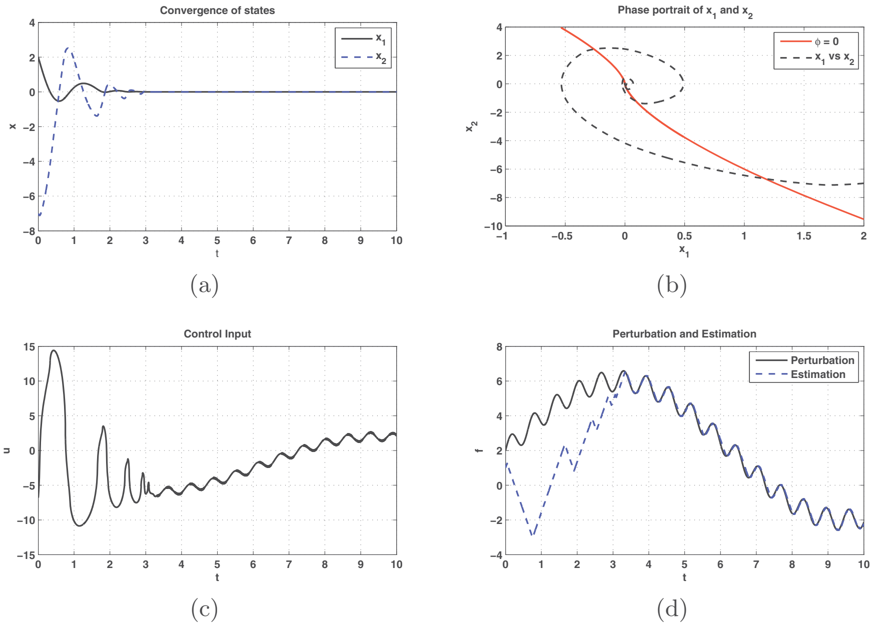


Fig. 7. Numerical example uncertain double integrator

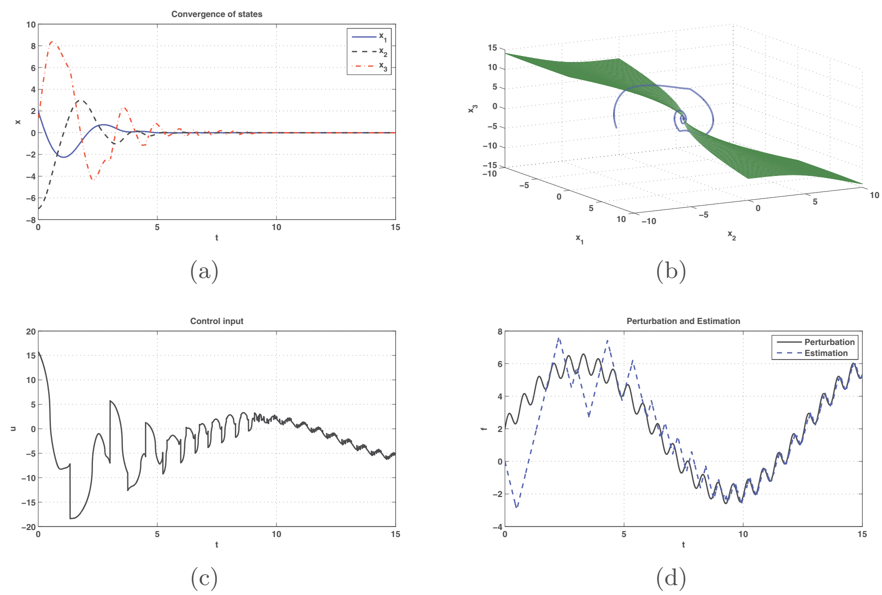


Fig. 8. Numerical example uncertain triple integrator

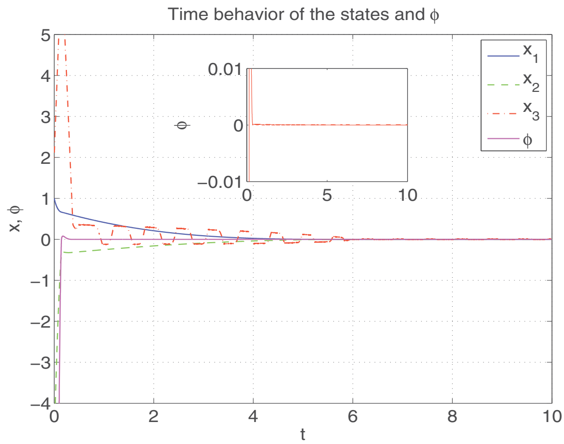


Fig. 9. Time evolution of the sates x_1, x_2, x_3 and of the switching variable $\phi = \phi_N$ under a non vanishing perturbation f

zoomed in the same picture. This behavior is similar to the one of the classical second-order sliding mode known as Terminal (or Prescribed) Controller.

Figure 13 presents again the phase portrait of the plant's states x_1 and x_2 with the same CNTSMC, but with different gains: $k_1 = 6$, $k_2 = 1/6$, $k_3 = 6$. Trajectories in Figure 13 have a rather undamped behavior compared to the ones in Figure 11. In this case, the convergence to the switching surface $\phi = \phi_N = 0$ has a twisting-like convergence to the switching surface (see Fig. 11 and Fig. 13).

6 Convergence Conditions for the Continuous Terminal Sliding Mode Algorithm

The proposed controllers (12) and (16) are able to stabilize system (2) in finite time if the following Proposition is satisfied.

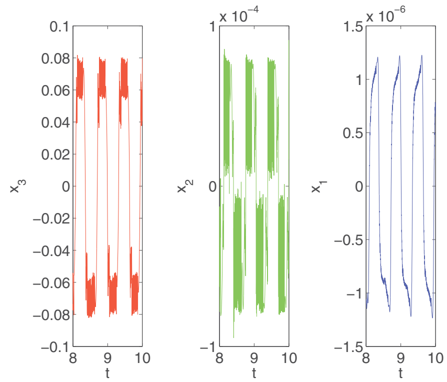
Proposition 1. *System* (14) *is finite time stable at the origin, with appropriate gains* k_1, k_2 *and* k_3*, in spite of bounded perturbations* $|\rho| \leq \Delta$.

of the bounded perturbation ρ.

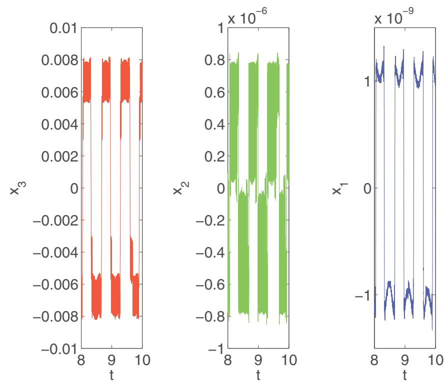
6.1 Lyapunov Function for Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA)

Consider the following continuous Lyapunov function candidate for the stability analysis of (14)

$$V(x) = p_1|x_1|^{\frac{4}{3}} - p_{12}\lfloor x_1\rceil^{\frac{2}{3}}\left(x_2 + k_2\lfloor x_1\rceil^{2/3}\right) + p_2\left|x_2 + k_2\lfloor x_1\rceil^{2/3}\right|^2 + p_{13}\lfloor x_1\rceil^{\frac{2}{3}}\lfloor x_3\rceil^2 - p_{23}\left(x_2 + k_2\lfloor x_1\rceil^{2/3}\right)\lfloor x_3\rceil^2 + p_3|x_3|^4. \quad (19)$$



(a)



(b)

Fig. 10. Precision of the state variables x_1, x_2, x_3 corresponding to a 3-order Sliding Mode

$V(x)$ is homogeneous of degree $\delta_V = 4$, with weights $\varrho = [3, 2, 1]$. It is differentiable everywhere, but it is not locally Lipschitz at $x_1 = 0$. Our main goal is to derive conditions for the coefficients $(p_1, p_{12}, p_2, p_{13}, p_{23}, p_3)$, and for the gains (k_1, k_2, k_3) of the continuous terminal sliding-mode algorithm (14), such that $V(x) > 0$ and time derivative of (19), along (14), is negative definite ($\dot{V} < 0$ for all $x \in \mathbb{R}^3, x \neq 0$).

Function (19) can also be expressed as a quadratic form, with the vector $\Xi^T = \left[\lfloor x_1 \rceil^{\frac{2}{3}} \quad \phi \quad \lfloor x_3 \rceil^2 \right]$, where $\phi = \left(x_2 + k_2 \lfloor x_1 \rceil^{2/3} \right)$, i.e.

$$V(x) = \Xi^T P \Xi, \quad \text{where} \quad P = \begin{bmatrix} p_1 & -\frac{1}{2}p_{12} & \frac{1}{2}p_{13} \\ -\frac{1}{2}p_{12} & p_2 & -\frac{1}{2}p_{23} \\ \frac{1}{2}p_{13} & -\frac{1}{2}p_{23} & p_3 \end{bmatrix} \tag{20}$$

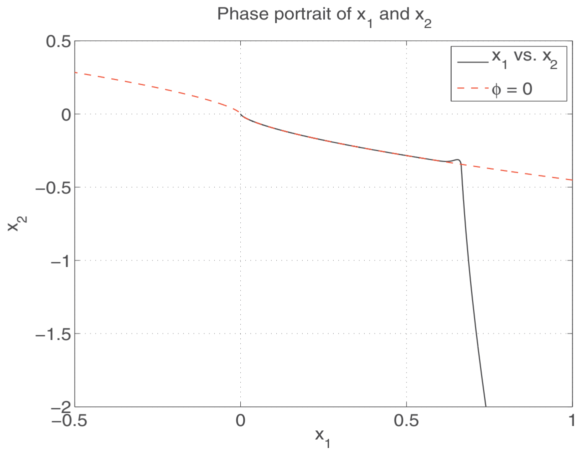


Fig. 11. Phase portrait of Plant's states x_1 and x_2, and locus of the switching curve $\phi = \phi_N = 0$, showing a Sliding-Like behavior of the CTSM controller

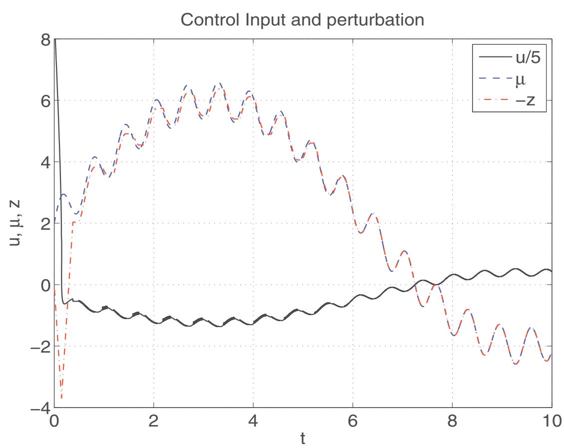


Fig. 12. Time behavior of the continuous control signal u_2, and (finite-time) estimation of the perturbation f by the controller state $-L = -z$

$V(x)$ is positive definite and radially unbounded if and only if $P > 0$, which is true if the following inequalities are satisfied

$$\left\{ \begin{array}{l} p_1 > 0, \quad p_1 p_2 > \frac{1}{4} p_{12}^2, \\ p_1 \left(p_2 p_3 - \frac{1}{4} p_{23}^2 \right) + \frac{p_{12}}{2} \left(-\frac{p_{12} p_3}{2} + \frac{p_{13} p_{23}}{4} \right) \\ + \frac{p_{13}}{2} \left(\frac{p_{12} p_{23}}{4} - \frac{p_2 p_{13}}{2} \right) > 0 \end{array} \right. \tag{21}$$

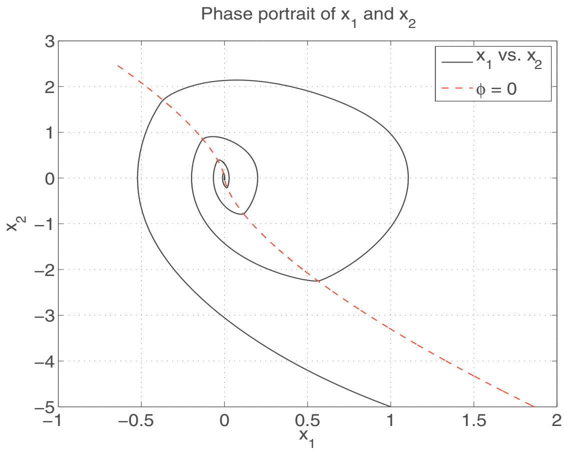


Fig. 13. Phase portrait of Plant's states x_1 and x_2, and locus of the switching curve $\phi = \phi_N = 0$, showing a Twisting-Like behavior of the CNTSM controller

The derivative of (19), along the system (14), is

$$\begin{aligned}\dot{V}(x) &= q_1\lfloor x_1\rceil^{\frac{1}{3}}x_2 - q_2|x_1|^{-\frac{1}{3}}x_2^2 - 2k_1p_2|\phi|^{\frac{3}{2}} - p_{23}|x_3|^3 \\ &- q_3|x_1|^{-\frac{1}{3}}x_2\lfloor x_3\rceil^2 + k_1p_{12}\lfloor x_1\rceil^{\frac{2}{3}}\lfloor\phi\rceil^{\frac{1}{2}} - \bar{q}_4\lfloor x_1\rceil^{\frac{2}{3}}x_3 \\ &+ \bar{q}_5x_3\phi + p_{23}k_1\lfloor\phi\rceil^{\frac{1}{2}}\lfloor x_3\rceil^2 - \bar{q}_6\lfloor x_3\rceil^3\lfloor\phi\rceil^0 \end{aligned} \tag{22}$$

where

$$\begin{cases} q_1 &= \frac{4p_1}{3} - \frac{4k_2p_{12}}{3} + \frac{4p_2k_2^2}{3} \\ q_2 &= \frac{2p_{12}}{3} - \frac{4p_2k_2}{3} \\ q_3 &= \frac{2p_{23}k_2}{3} - \frac{2p_{13}}{3} \\ \bar{q}_4 &= p_{12} + 2p_{13}k_3\lfloor\phi\rceil^0\lfloor x_3\rceil^0 - 2p_{13}\lfloor x_3\rceil^0\rho \\ \bar{q}_5 &= 2p_2 + 2p_{23}k_3\lfloor\phi\rceil^0\lfloor x_3\rceil^0 - 2p_{23}\lfloor x_3\rceil^0\rho \\ \bar{q}_6 &= 4k_3p_3 - 4p_3\rho\lfloor\phi\rceil^0 \end{cases} \tag{23}$$

when $\rho = 0$, then let us define

$$\begin{cases} q_4 &= p_{12} + 2p_{13}k_3\lfloor\phi\rceil^0\lfloor x_3\rceil^0 \\ q_5 &= 2p_2 + 2p_{23}k_3\lfloor\phi\rceil^0\lfloor x_3\rceil^0 \\ q_6 &= 4k_3p_3 \end{cases} \tag{24}$$

The following Definition and Lemma are presented to prove the stability of system (14) without disturbances i.e., $\rho = 0$

Definition 1. *Functions $\beta(\alpha,\lambda)$ and $\vartheta(\alpha)$ are the real valued function of the real variable $\alpha > 0$ and any value of λ, $\beta(\alpha,\lambda)$ satisfied $\vartheta(\alpha) \geq \beta(\alpha,\lambda)$ for all λ, where the function $\beta(\alpha,\lambda)$ is defined as*

$$\beta(\alpha,\lambda) = \begin{cases} max(0, \beta_1(\alpha,\lambda)) & for \;\; \lambda \geq -\sqrt{3\alpha} \\ max(0, \beta_2(\alpha,\lambda)) & for \;\; \lambda < -\sqrt{3\alpha} \end{cases} \tag{25}$$

where

$$\begin{cases} \beta_1(\alpha,\lambda) & = -\alpha r_1^3(\alpha,\lambda) + \lambda r_1^2(\alpha,\lambda) + r_1(\alpha,\lambda) \\ \beta_2(\alpha,\lambda) & = \alpha r_2^3(\alpha,\lambda) - \lambda r_2^2(\alpha,\lambda) + r_2(\alpha,\lambda) \end{cases} \tag{26}$$

and

$$r_1(\alpha,\lambda) = \frac{\lambda + \sqrt{|\lambda|^2 + 3\alpha}}{3\alpha}, r_2(\alpha,\lambda) = \frac{\lambda - \sqrt{|\lambda|^2 - 3\alpha}}{3\alpha} \tag{27}$$

One of the main results of the chapter which guarantee the finite time stability of proposed algorithm (14) when $\rho = 0$ is stated in the following lemma :

Lemma 1. *Consider the continuous and homogeneous function $V(x)$ given by (20). $V(x)$ goes to zero in finite time if the following conditions are satisfied*

$$\begin{cases} p_1 + p_2 k_2^2 > k_2 p_{12}, \;\; p_{12} = 2p_2 k_2, \;\; p_{23} k_2 = p_{13} \\ p_{12} > 2p_{13} k_3, \;\; p_2 > p_{23} k_3, \;\; k_3 > 0, \end{cases} \tag{28}$$

and there exists some $\alpha_1, \alpha_2 > 0$ such that

$$\begin{cases} q_1 k_1 k_2 p_{12} - k_1 p_{12} - \sqrt{\frac{2^2 |q_4|^3}{3^3 (p_{23} - |q_6|)}} > \alpha_1 > 0 \\ \vartheta(\alpha_1) \geq \beta(q_1, \alpha_1) \\ \frac{2k_1^2 p_2 p_{23} - \alpha_2}{k_1^3 p_{23} p_{12}} > \vartheta(\alpha_1) > 0, \end{cases} \tag{29}$$

$$\begin{cases} 2k_1^2 p_2 p_{23} > \alpha_2 > 0 \\ \vartheta(\alpha_2) \geq max\{\beta(\lambda_1, \alpha_2), \beta(\lambda_2, \alpha_2)\} \\ \frac{1}{(k_1 p_{12})^2} \left(p_{23} - |q_6| - \frac{2^2 |q_4|^3}{3^3 \left(q_1 k_2 - \frac{\alpha_1}{k_1 p_{12}} \right)^2} \right) > \vartheta(\alpha_2) > 0, \end{cases} \tag{30}$$

where $\lambda_1 = 2p_2 + 2p_{23} k_3$ and $\lambda_2 = 2p_2 - 2p_{23} k_3$. In this case $V(x)$ satisfies the differential inequality

$$\dot{V} \leq -\kappa V^{3/4} \tag{31}$$

for some positive κ and it is a Lyapunov function for the system (14), whose trajectories converges in finite time to the origin $x = 0$, for every value of the perturbation $\rho = 0$. The convergence time of a trajectory starting at the initial condition x_0 can be estimated as

$$T(x_0) \leq \frac{4}{\kappa} V^{\frac{1}{4}}(x_0). \tag{32}$$

Lemma 1 provides conditions for the existence of a Lyapunov function for system (14), when $\rho = 0$. However, it is not obvious a priori that there exist indeed values of the parameters k_1, k_2, k_3,p_1, p_2, p_3,p_{12}, p_{13}, p_{23}, α_1, α_2 for which the conditions imposed in the Lemma 1 are satisfied, i.e. if the system of inequalities are feasible. Using the next Theorem it will be shown that there are indeed sets of values for the parameters, that fulfill the conditions of the Theorem in the both case when $\rho = 0$ or $\rho \neq 0$. This Theorem is the main contribution of the chapter, which also gives the proof of the Proposition 1.

Theorem 1. *Let us suppose that the origin $x = 0$ of system (14) is finite time stable for a set of gains k_1, k_2, k_3, and (19) is the Lyapunov function $V(x)$, with a set of parameters $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}$ in the unperturbed case. Then, the origin $x = 0$ of (14) remains finite time stable for a set of gains l^3k_1, l^2k_2, l^6k_3 and that $V(x)$ in (19) is a Lyapunov function for the set of parameters $l^{-8}p_1, l^{-12}p_2, l^{-24}p_3, l^{-10}p_{12}, l^{-16}p_{13}, l^{-18}p_{23}$, for the sufficiently large positive real number l in the both perturbed and unperturbed case.*

Table 1. Parameters of the Lyapunov function when $\rho = 0$

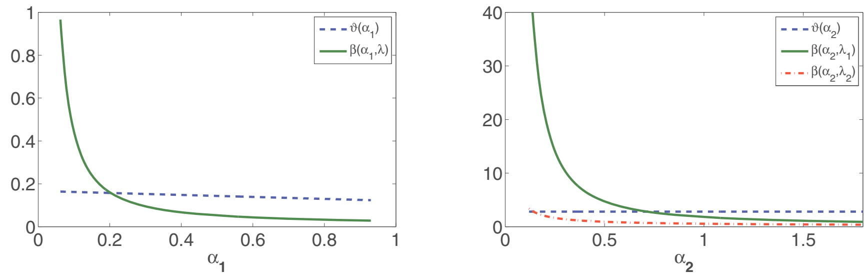
k_1	6	6
k_2	1	2
k_3	6	6
p_1	20	20
p_2	0.5	0.5
p_3	0.01	0.01
p_{12}	1	2
p_{13}	0.05	0.1
p_{23}	0.05	0.05

6.1.1 Lyapunov Function Validation

After finding the conditions on the gains k_1, k_2, k_3, as given by (28), based on the Lyapunov function parameters of (20), $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}$ that makes system (14) finite time stable at the origin, it is still not quite obvious that system inequalities (21) and (28) to (30) are feasible. Therefore, we have to find at least one set of numerical values of k_1, k_2, k_3, $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}$ and inequalities for α_1 and α_2 such that inequalities (21) and (28) to (30) are feasible. Other sets of gains can easily be found using Theorem 1, by simply tuning the positive real value l.

One of the Possible Choice for Validation:

Using (28) to (30) and some particular $k_i, i = 1, 2, 3$ and $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}$, one can write $\alpha_2 = \eta_1\alpha_1$ where $0 < \eta_1 < 1$. Figures 14 (a) and (b) show the graphs of the functions $\vartheta(\alpha_1)$, $\beta(\alpha_1, \lambda)$ and $\vartheta(\alpha_2)$, $\beta(\alpha_2, \lambda_1)$ along with $\beta(\alpha_2, \lambda_2)$, respectively, for the parameters of the first column of Table 1 parameters which satisfy (29) and (30).



(a) Graph of $\vartheta(\alpha_1)$, $\beta(\alpha_1, \lambda)$ (b) Graph of $\vartheta(\alpha_2)$, $\beta(\alpha_2, \lambda_1)$ and $\beta(\alpha_2, \lambda_2)$

Fig. 14. Inequalities for the Lyapunov Function Validation

6.2 Lyapunov Analysis of Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA)

The Lyapunov function candidate for system (18) is proposed as

$$V(x) = \beta |x_1|^{\frac{5}{3}} + x_1 x_2 + \frac{2}{5} k_2 |x_2|^{\frac{5}{2}} - \frac{1}{k_1^3} x_2 x_3^3 + \gamma_3 |x_3|^5 ,$$

which is homogeneous (of degree $\delta_V = 5$) and continuously differentiable. We will show that $V(x)$ is decrescent, and that selecting $\beta > 0$ and $\gamma_3 > 0$ sufficiently large it is also positive definite.

For this, recall the classical Young's inequality [16]: for any real values $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and any positive real numbers a, b, c the inequality $ab \leq c^p \frac{a^p}{p} + c^{-q} \frac{b^q}{q}$ holds. Using this inequality it follows that

$$V(x) \geq \left(\beta - \frac{3}{5} c_1^{\frac{5}{3}}\right) |x_1|^{\frac{5}{3}} + \frac{2}{5}\left(k_2 - c_1^{-\frac{5}{2}} - c_2^{-\frac{5}{2}} \frac{1}{k_1^3}\right) |x_2|^{\frac{5}{2}} + \left(\gamma_3 - \frac{3}{5} \frac{1}{k_1^3} c_2^{\frac{5}{3}}\right) |x_3|^5 .$$

V is positive definite if all its coefficients are positive. This can be achieved by selecting e.g. $c_1 = \left(\frac{4}{k_2}\right)^{\frac{2}{5}}$, $c_2 = \left(\frac{4}{k_2 k_1^3}\right)^{\frac{2}{5}}$, and

$$\beta > \frac{3}{5}\left(\frac{4}{k_2}\right)^{\frac{2}{3}} , \tag{33}$$

$$k_1^5 \gamma_3 > \frac{3}{5}\left(\frac{4}{k_2}\right)^{\frac{2}{3}} . \tag{34}$$

It is straight forward to verify that V is decrescent for any values of the parameters if the following conditions are satisfied

$$
\begin{aligned}
k_3 &> \Delta \\
\beta &> \frac{3}{5k_2^{\frac{2}{3}}} \\
k_1 &> 5\gamma_3 k_1^5 \kappa \left(1+\frac{\Delta}{k_3}\right) + \frac{\left(\frac{3}{2}\right)^2 \left(\frac{k_3}{k_1}\right)^2}{\left(\frac{5}{3}\beta k_2^{\frac{2}{3}}-1\right)} \left(1+\frac{\Delta}{k_3}\right)^2 \\
\gamma_3 k_1^5 &> \frac{\left(\frac{5}{3}2^{\frac{1}{3}}\beta + 3\frac{k_3}{k_1}\left(1+\frac{\Delta}{k_3}\right)\right)^2}{20\frac{k_3}{k_1}\left(1-\frac{\Delta}{k_3}\right)\left(\frac{5}{3}\beta k_2^{\frac{2}{3}}-1\right)}
\end{aligned}
\tag{35}
$$

7 Continuous Nested Sliding Mode Algorithm

In this section a generalization of the Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA) is presented. Due to the nested structure of the algorithm, it is also referred to as continuous nested terminal sliding-mode algorithm.

CSTSMA is proposed as follows

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 \left|\phi_1\right|^{1/2} \operatorname{sign}\left(\phi_1\right) + x_3 \\
\dot{x}_3 &= -k_3 \operatorname{sign}\left(\phi_1\right) + \rho
\end{aligned}
\tag{36}
$$

where $\phi_1 = x_2 + k_2|x_1|^{2/3}\operatorname{sign}(x_1)$ $x_1, x_2,$ x_3 represent the states, and the perturbation ρ satisfies $|\rho| \leq \Delta$.

4-CSNSMA is proposed as follows

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -k_1 \left|\phi_2\right|^{1/2} \operatorname{sign}\left(\phi_2\right) + x_4 \\
\dot{x}_4 &= -k_4 \operatorname{sign}\left(\phi_2\right) + \rho
\end{aligned}
\tag{37}
$$

where

$$
\phi_2 = x_3 + k_3 \left(|x_1|^3 + |x_2|^4\right)^{\frac{1}{6}} \operatorname{sign}\left(x_2 + k_2|x_1|^{\frac{3}{4}}\operatorname{sign}(x_1)\right) \tag{38}
$$

and $x_1, x_2,$ $x_3,$ x_4 represent the states, and the perturbation ρ satisfies $|\rho| \leq \Delta$.

Similarly, 5-CSNSMA is proposed as follows

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -k_1 \left|\phi_3\right|^{1/2} \operatorname{sign}\left(\phi_3\right) + x_5 \\
\dot{x}_5 &= -k_5 \operatorname{sign}\left(\phi_3\right) + \rho
\end{aligned} \tag{39}$$

where

$$\phi_3 = x_4 + k_4 \left[\left(|x_1|^{12} + |x_2|^{15} + |x_3|^{20} \right)^{\frac{1}{30}} \operatorname{sign}\left(l_1\right) \right]$$

and

$$l_1 = x_3 + k_3 \left(|x_1|^{12} + |x_2|^{15} \right)^{\frac{1}{20}} \operatorname{sign}\left(x_2 + k_2 |x_1|^{\frac{4}{5}} \operatorname{sign}(x_1) \right)$$

and $x_1, x_2,\ x_3,\ x_4, x_5$ represent the states, and the perturbation ρ satisfies $|\rho| \leq \varDelta$.

The generalized r-CSNSMA is proposed as follows

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{r-1} &= -k_1 \left|\phi_{r-2}\right|^{1/2} \operatorname{sign}\left(\phi_{r-2}\right) + x_r \\
\dot{x}_r &= -k_r \operatorname{sign}\left(\phi_{r-2}\right) + \rho
\end{aligned} \tag{40}$$

where $x_1, x_2, \cdots, x_r$ represent the states, and the perturbation ρ satisfies $|\rho| \leq \varDelta$. Variable ϕ_{r-2} is defined as:

- $$R_{1,r-1} = |x_1|^{\frac{r}{r+1}}$$

  where r represents the relative degree of the algorithm with respect to x_1.
- $$R_{i,r-1} = \left||x_1|^{r_1} + |x_2|^{r_2} + \cdots + |x_{i-2}|^{r_{i-2}}\right|^{q_i},$$

  where $i = 2, 3, \cdots, (r-1)$, $r_1, r_2, \cdots, r_{i-2}$, and q_i is a parameter designed based on the homogeneity weight of x_{i+1}.
- $$S_{0,r-1} = x_1$$

  $$S_{1,r-1} = x_2 + k_2 R_{1,r-1} \operatorname{sign}(x_1)$$

  $$S_{i,r-1} = x_{i+1} + k_{i+1} R_{i,r-1} \operatorname{sign}(S_{i-1,r-1})$$

  where $i = 2, 3, \cdots, (r-1)$
- Finally $\phi_{r-2} = s_{r-1,r-1}$.

For example, if we want to select r_1, r_2 and q_2 for the 4-CSNSMA (r=3), firstly we have to check its weighted homogeneity. Our aim is to design a 4-CSNSMA that has homogeneous weights $\{4, 3, 2, 1\}$. The design of parameters r_1, r_2, and q_2 is fully dependent on the weight assigned to x_3. Here, we have chosen its value as 2, therefore, it is necessary to adjust r_1, r_2 and q_2 such that after homogenous scaling, one can get the desired value 2 for x_3. There are several ways to select these parameters, one of them consists on calculating the LCM (lowest common factor) of 4 and 3, which is 12, and then adjusting the power of the terms $|x_1|$, and $|x_2|$, such that the weight of x_3 is equal to 2. It is obvious that by selecting $r_1 = 3, r_2 = 4$ and $q_2 = 6$ it is possible to maintain the homogeneity of the algorithm with weights $\{4, 3, 2, 1\}$. Similarly, one can generalize the 4-CSNSMA till the r-CSNSMA.

Evolution of the states for the STA, where STA is given as follows [19], [28]

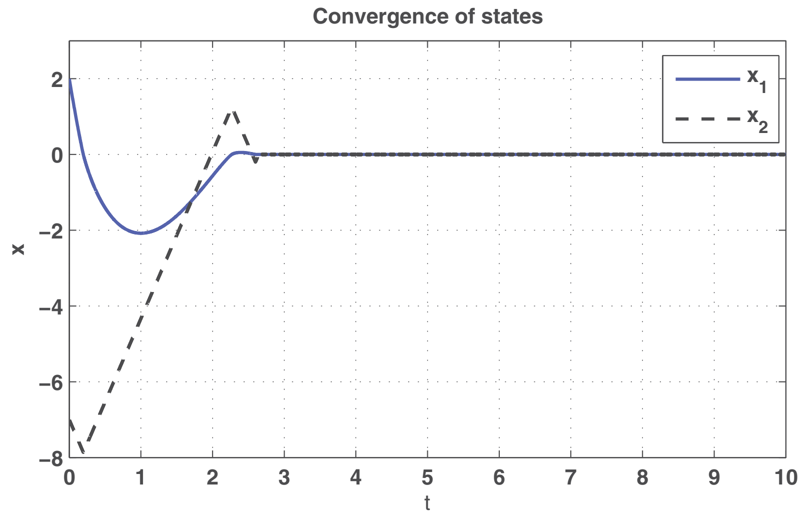
$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{\frac{1}{2}}\text{sign}(x_1) + x_2 \\ \dot{x}_2 &= -k_2\text{sign}\,(x_1) + \rho, \end{aligned} \tag{41}$$

where x_1, x_2 represent the states, and the perturbation ρ satisfies $|\rho| \leq \Delta$. The CSTSMA, and 4-CSNSMA are shown in Fig. 15-Fig. 16, with the following values for the initial conditions and gains

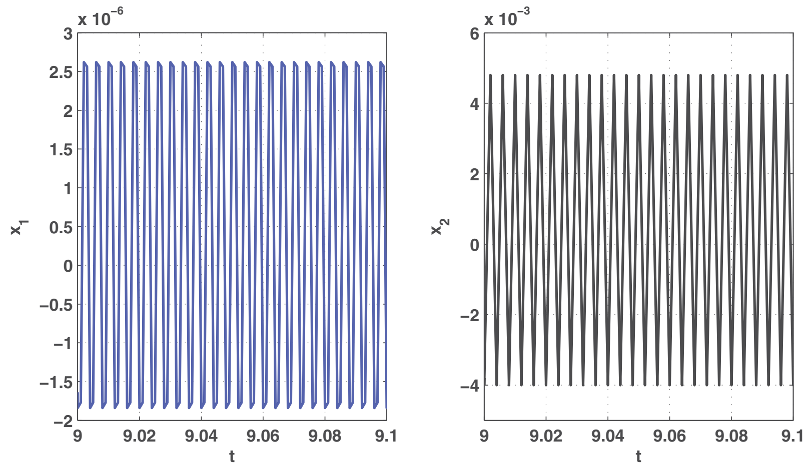
- STA
  - initial conditions $x_1(0) = 2$, $x_2(0) = -7$
  - gains $k_1 = 3, k_2 = 4$
- CSTSMA
  - initial conditions $x_1(0) = 2$, $x_2(0) = -7$ and $x_3(0) = 1$
  - gains $k_1 = 6, k_2 = 2$ and $k_3 = 6$
- 4-CSNSMA
  - initial conditions $x_1(0) = 2, x_2(0) = -7$, $x_3(0) = 1$ and $x_4(0) = -1$
  - gains $k_1 = 4, k_2 = 1, k_3 = 2$ and $k_4 = 4$

Remark 1. The properties of the proposed algorithms are the same as those of the terminal sliding mode, therefore it is referred to as r^{th} order continuous terminal sliding-mode algorithm (r-CSNSMA).

Discussion about CTSMA and Other Generalized CSNSMA. The CTSMA (36) is homogeneous of degree $\delta_f = -1$ with weights $\varrho = \{3, 2, 1\}$. The main advantage of this algorithm is that the only information that it needs to maintain finite time convergence of all three variables x_1, x_2 and x_3 is the output (x_1) and its derivative (x_2). The proposed algorithm can work as a controller for an uncertain system with relative degree 2 with respect to its output, in the case of CSTSMA. Similarly, the r-CSNSMA is homogeneous of degree $\delta_f = -1$, with weights $\varrho = \{r, r-1, \cdots, 2, 1\}$, and it can be used for an uncertain system with relative degree $r-1$ with respect to output. The main idea behind the construction of this algorithm is to add one extra discontinuous integral term, which is able to reconstruct the perturbation and also nullify it. The CSNSMA is insensible to perturbations whose time derivative is bounded. These perturbations



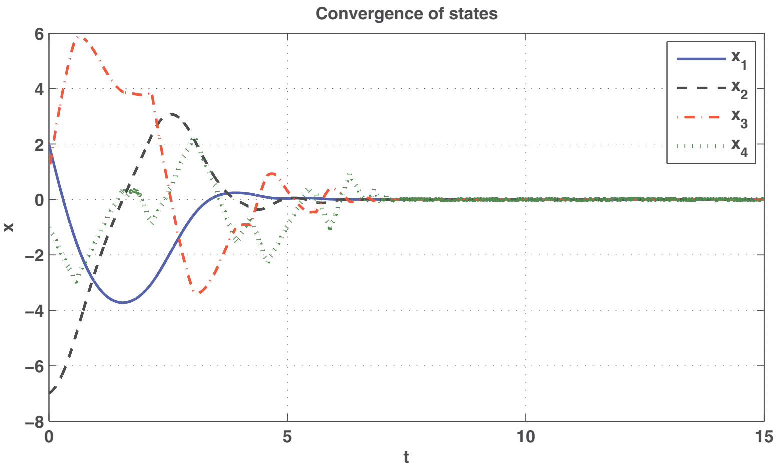
(a)



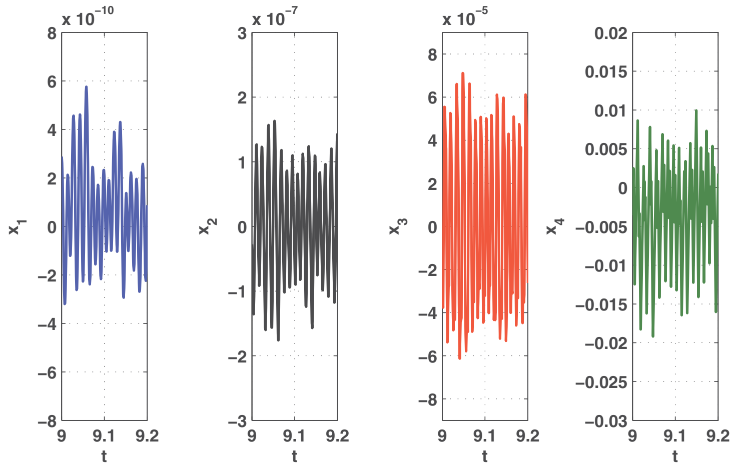
(b)

Fig. 15. STA and states precision with $\tau = 0.001$

could grow no more fast than linear function of time, i.e., they do not need to be bounded. In comparison the nested HOSM controller can compensate bounded Lebesgue measurable perturbations. Comparison of the properties of the principal SMC strategies for the second order uncertain system is given in the Table 2.



(a)



(b)

Fig. 16. CSTSMA and states precision with $\tau = 0.001$

Simulation Results. In order to verify the proposed technique of the r-CSNSMA, the following second and third order systems are considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + f \end{aligned} \tag{42}$$

Table 2. Comparison of the properties of the different SMC strategies for the second order uncertain system with output σ

Algorithm	Control Signal	Information	Stability	Chattering	Order of sliding w.r.t. σ
First SMC	Discontinuous	$\sigma, \dot{\sigma}$	Asymptotic	Yes	1
Twisting	Discontinuous	$\sigma, \dot{\sigma}$	Finite time	Yes	2
Terminal SMC	Discontinuous	$\sigma, \dot{\sigma}$	Finite time	Yes	2
STC	Continuous	$\sigma, \dot{\sigma}$	Asymptotic	No	2
Third SMC	Continuous	$\sigma, \dot{\sigma}, \ddot{\sigma}$	Finite time	No	3
Continuous Terminal SMC	**Continuous**	$\sigma, \dot{\sigma}$	**Finite time**	**No**	3

where x_1, x_2 are the states, u is the control and $f = 2 + 4sin(t/2) + 0.6sin(10t)$ is the Lipschitz (in time) disturbance. Similarly,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u + f \end{aligned} \tag{43}$$

where x_1, x_2, x_3 are the states, u is the control and $f = 2+4sin(t/2)+0.6sin(10t)$ is the Lipschitz (in time) disturbance. The controller for systems (42) and (43) are designed as

$$u = -k_1 \left|\phi_1\right|^{1/2} \text{sign}\left(\phi_1\right) - \int_0^t k_3 \text{sign}\left(\phi_1\right) d\tau \tag{44}$$

and

$$u = -k_1 \left|\phi_2\right|^{1/2} \text{sign}\left(\phi_1\right) - \int_0^t k_4 \text{sign}\left(\phi_2\right) d\tau \tag{45}$$

where ϕ_1 and ϕ_2 are defined as in (14) and (37), respectively. The following parameters are used for the simulation

- uncertain double order integrator (42)
  - initial conditions $x_1(0) = 2$ and $x_2(0) = -7$
  - gains $k_1 = 6, k_2 = 5$ and $k_3 = 6$
- uncertain third order integrator (43)
  - initial conditions $x_1(0) = 2, x_2(0) = -7$ and $x_3(0) = 1$
  - gains $k_1 = 5, k_2 = 1, k_3 = 2$ and $k_4 = 4$

How to Implement CSNSMA? The main specific feature of CSNSMA as well as of STA, CSTSMA and CNTSMA is that the part of control signal responsable for the compensation of Lipschitz perturbation is continuous. As a consequence of this, they are only able to compensate, theoretically exactly, Lipschitz perturbations, but they also need a Lipschitz signal to follow. This means that the switching surfaces estimated by the differentiators should be smooth and have Lipschitz derivatives. This is the reason why even the CSNSMA requires only $(r-1)$ derivatives of the sliding output, and the derivative of order $(r-1)$

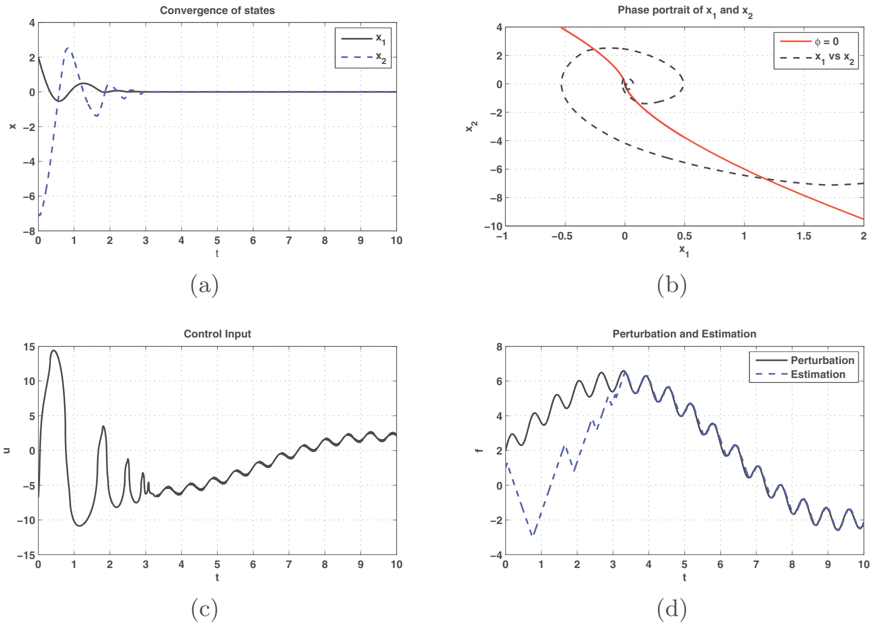


Fig. 17. Numerical example uncertain double integrator

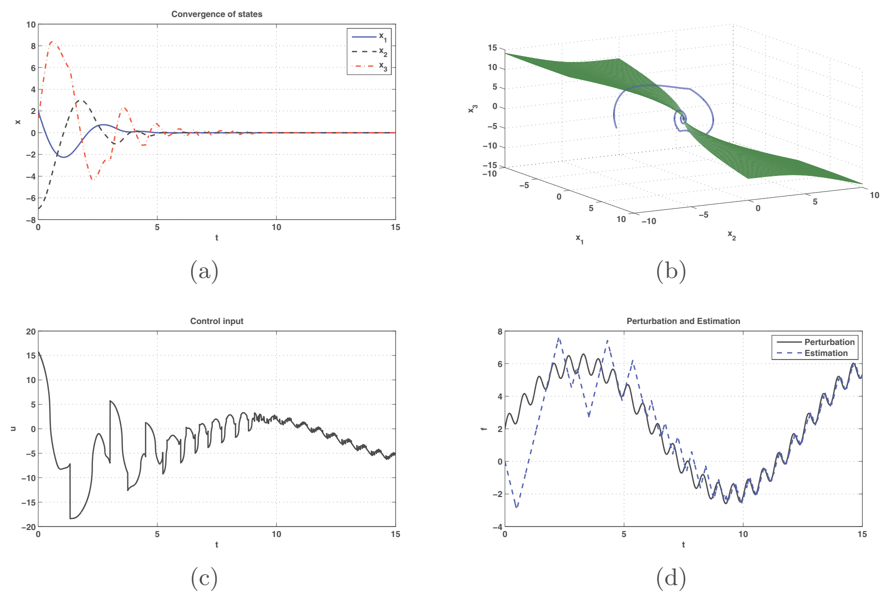


Fig. 18. Numerical example uncertain triple integrator

should be approximated by smooth signal with Lipschitz derivatives. This could be achieved by a differentiator of order r so, in order to implement the CSNSMA it is necessary to use r-order robust exact differentiators [22], but using only the derivatives that it produces up to the order $(r-1)$. By doing so, all the necessary signals for the r-CSNSMA will be smooth, with Lipschitz derivative. The block diagram for the implementation of CSNSMA is shown in next Figure.

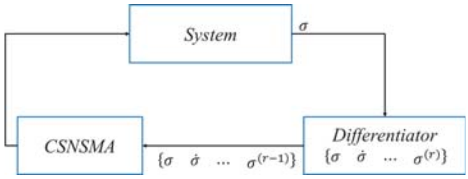


Fig. 19. CSNSMA implementation

8 Conclusion

In this chapter, the historical overview of the development of SMC is presented. We have shown that in the last three decades the Sliding Mode Community has created new generations of controllers:

- second order sliding mode controllers(1985);
- super-twisting controllers(1993);
- arbitrary sliding-mode controllers(2001,2005).

In this chapter we have presented the next generation: two families of continuous nested sliding-mode controllers, that can be used on Lipschitz systems with relative degree r, providing the continuous control signal. This new controllers ensure a finite-time convergence of the sliding output to the $(r+1)-th$-order sliding set using information on the sliding output and its derivatives up to the order $(r-1)$,

Anknowledgements. With this chapter the authors want to acknowledge the contribution of Professor Okyay Kaynak in the development of sliding mode control. His energy, his willing and abilities to help to all of colleagues, the excellent organization VSS08 workshop pushing the VSS community to reach the high level results. **Thank you Okyay!**

The authors also want to acknowledge to the joint scientific program Mexico-India, CONACyT- DST/INT/MEX/RPO-02/2008, the CONACyT grant 132125, PAPIIT grants 113613 and 113614 of UNAM and Fondo de Colaboracion II-FI the grant IIGSBAS-122-2014.

References

1. Baccioti, A., Rosier, L.: Lyapunov functions and stability in control theory, 2nd edn. Springer, New York (2005)
2. Bartolini, G., Ferrara, A., Usai, E.: Chattering avoidance by second-order sliding-mode control. IEEE Transactions on Automatic Control 43(2), 241–246 (1998)
3. Bondarev, A.G., Bondarev, S.A., Kostyleva, N.E., Utkin, V.I.: Sliding modes in systems with asymptotic state observers. Automation and Remote Control 5, 5–11 (1985)
4. Boiko, I., Fridman, L.: Analysis of chattering in continuous sliding mode controllers. IEEE Transaction on Automatic Control 50(11), 1442–1446 (2005)
5. Boiko, I., Fridman, L., Pisano, A., Usai, E.: Analysis of chattering in systems with second-order sliding modes. IEEE Transactions on Automatic Control 52(11), 2085–2102 (2007)
6. Chalanga, A., Kamal, S., Fridman, L., Bandyopadhyay, B., Moreno, J.A.: How to implement Super-Twisting Controller based on sliding mode observer? In: Proc. of 13th IEEE Workshop on Variable Structure Systems, Nantes (2014), doi:10.1109/VSS.2014.6881145
7. Drazenovic, B.: The invariance conditions in variable structure systems. Automatica 5(3), 287–295 (1969)
8. Edwards, C., Spurgeon, S.: Sliding Mode Control. Taylor and Francis, London (1998)
9. Efimov, D.: Global sliding-mode observer with adjusted gains for locally Lipschitz systems. Automatica 47(3), 565–570 (2011)
10. Emel'yanov, S.V., Korovin, S.K., Levantovski, L.V.: Higher-order sliding modes in binary control systems. Sov. Phys. Dokl 31(4), 291–293 (1986)
11. Feng, Y., Yu, X., Man, Z.: Non-singular terminal sliding mode control of rigid manipulators. Automatica 38(12), 2159–2167 (2002)
12. Filippov, A.F.: Diffrential equations and discontinuous right hand side. Kluwer, Dordrecht, The Netherlands (1988)
13. Fridman, L.: Sliding mode enforcement after 1990: Main results and some open problems. In: Fridman, L., Moreno, J., Iriarte, R. (eds.) Sliding Modes. LNCIS, vol. 412, pp. 3–57. Springer, Heidelberg (2011)
14. Fridman, L.: On robustness of sliding mode systems with discontinuous control function. Automatika i Telemechanica (Automation & Remote Control 5, 172–175 (1985)
15. Gonzalez, T., Moreno, J., Fridman, L.: Variable Gain Super-Twisting Sliding Mode Control. IEEE Transactions on Automatic Control 57(8), 2100–2105 (2012)
16. Hardy, G.H., Littlewood, J.E., Plya, G.: Inequalities, 2nd edn. Cambridge University Press, London (1988)
17. Kamal, S., Chalanga, A., Moreno, J.A., Fridman, L., Bandyopadhyay, B.: Higher order super-twisting algorithm. In: Proc. of 13th IEEE Workshop on Variable Structure Systems, Nantes (2014), doi:10.1109/VSS.2014.6881129
18. Kolmogorov, A.N.: On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval. Amer. Math. Soc. Transl. 2(9), 233–242 (1962)
19. Levant, A.: Sliding order and sliding accuracy in sliding mode control. International Journal of Control 58(6), 1247–1263 (1993)
20. Levant, A.: Robust exact differentiation via sliding mode technique. Autmatica 34(3), 379–384 (1998)

21. Levant, A.: Universal single-input-single-output (SISO) sliding-mode controllers with finite-time convergence. IEEE Transactions on Automatic Control 6(9), 1447–1451 (2001)
22. Levant, A.: Higher-order sliding modes,differentiation and output-feedback control. Int. J. Control 76(9), 924–941 (2003)
23. Levant, A.: Homogeneity approach to high-order sliding mode design. Automatica 41(5), 823–830 (2005)
24. Levant, A.: Principles of 2-sliding mode design. Automatica 43(4), 576–586 (2007)
25. Levant, A.: Chattering analysis. IEEE Transaction on Automatic Control 55(6), 1380–1389 (2010)
26. Levant, A., Fridman, L.: Accuracy of homogeneous Sliding Modes in the Presence of Fast Actuators. IEEE Transactions on Automatic Control 55(3), 810–814 (2010)
27. Man, Z., Paplinski, A.P., Wu, H.R.: A robust MIMO terminal sliding mode control scheme for rigid robotic manipulators. IEEE Transaction on Automatic Control 39(12), 2464–2469 (1994)
28. Moreno, J.A., Osorio, M.: Strict Lyapunov functions for the Super-twsiting algorithm. IEEE Transactions on Automatic Control 57(4), 1035–1040 (2012)
29. Moreno, J.A.: Lyapunov function for Levant's second order differentiator. In: Proceedings of the 51st IEEE Conference on Decision and Control Maui, Hawaii, December 10–13 (2012)
30. Orlov, Y.: Discontinuous systems: Lyapunov analysis and robust synthesis under uncertainty conditions. Springer (2008)
31. Orlov, Y., Aoustin, Y., Chevallereau, C.: Finite time stabilization of a perturbed double integrator—Part I: Continuous sliding mode-based output feedback synthesis. IEEE Transactions on Automatic Control 56(3), 614–618 (2011)
32. Polyakov, A., Poznyak, A.: Reaching time estimation for super-twisting second order sliding mode controller via Lyapunov function designing. IEEE Transactions on Automatic Control 54(8), 1951–1955 (2009)
33. Polyakov, A., Poznyak, A.: Unified Lyapunov function for a finite–time stability analysis of relay second–order sliding mode control systems. IMA Journal of Mathematical Control and Information 29(4), 529–550 (2012)
34. Sanchez, T., Moreno, J.A.: Lyapunov functions for Twisting and Terminal controllers. In: Proceedings of the 13th International Workshop on Variable Structure Systems Nantes, France (2014)
35. Utkin, V.: Sliding modes in control and optimization. Springer (1992); (translated from Russian edition 1981)
36. Utkin, V.: Variable structure systems- Present and future. Automation and Remote Control 44(9), 1105–1120 (1984)
37. Utkin, V.: First Stage of VSS: people and events. In: Yu, X., Xu, J.-X. (eds.) Variable Structure Systems: Towards the 21st Century. LNCS, vol. 274, pp. 1–32. Springer, Heidelberg (2002)
38. Yu, X., Man, Z.: Fast terminal sliding mode control design for nonlinear dynamic systems. IEEE Transactions on Circuits and Systems Part I 39(2), 261–264 (2002)
39. Yu, X., Man, Z.: Model reference adaptive control systems with terminal sliding modes. International Journal of Control 64(6), 1165–1176 (1996)