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Prof. B. Bandyopadhyay

FNAE, IEEE Fellow, Institute Chair Professor and Head
Systems and Control Engineering Group
Indian Institute of Technology Bombay
India

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Continuous Integral Sliding Mode Control:
A Second Order Sliding Mode Approach

Outline

1. Introduction
2. Finite-time control
3. Integral sliding mode
4. Continuous integral sliding mode control
5. Super-twisting controller based on super-twisting observer
6. Super Twisting Output Feedback Control
7. Higher order sliding mode observer based super twisting control

- Discontinuity in feedback control is not suitable for practical applications, as a consequence of chattering.
- Replacing the discontinuous part with STA is an option.
- Replacement is possible due to unique disturbance observation property of STA.

sign(x) as a Disturbance Observer

$$\dot{x} = u + d, \tag{1}$$

where $x \in \mathbb{R}$, $u = -k\text{sign}(x) \in \mathbb{R}$, $d \in \mathbb{R}$ and $k > |d|_{\max}$.

It is shown that when $x = 0$, the equivalent value of control is obtained by setting $\dot{x} = 0$, which implies $d = [k\text{sign}(x)]_{eq}$.

Super Twisting as a Disturbance Observer

$$\begin{aligned} u &= -\lambda|x|^{\frac{1}{2}}\text{sign}(x) + v \\ \dot{v} &= -\alpha\text{sign}(x) \end{aligned} \tag{2}$$

After substituting the control input (2) into (1), one can write

$$\begin{aligned} \dot{x} &= -\lambda|x|^{\frac{1}{2}}\text{sign}(x) + v + d \\ \dot{v} &= -\alpha\text{sign}(x) \end{aligned} \tag{3}$$

Let us define $z := v + d$, then $\dot{z} = \dot{v} + \dot{d}$

The system can be written as

$$\begin{aligned} \dot{x} &= -\lambda |x|^{\frac{1}{2}} \mathrm{sign}(x) + z \\ \dot{z} &= -\alpha \mathrm{sign}(x) + \dot{d}, \qquad |\dot{d}| \leq \Delta, \end{aligned} \tag{4}$$

If $\alpha = 1.1\Delta$ and $\lambda = 1.5\sqrt{\Delta}$, it has been proved that $x = z = 0$ in finite time.

$z = 0$ implies $d = -v$, so one can also construct the disturbance.

The disturbance is given by

$$\begin{aligned} d &= -v \\ \dot{v} &= -\alpha \mathrm{sign}(x) \end{aligned} \tag{5}$$

when x and $v + d$ are zero in (3).

Finite Time Stabilization of Chain of Integrators

Consider the following chain of integrators

$$\begin{aligned}\dot{x}_1 &= x_2\\ &\vdots\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= u\end{aligned} \tag{6}$$

Theorem : Bhat and Bernstein

Let $k_1, \ldots, k_n > 0$ be such that the polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$ is Hurwitz, and there exists $\epsilon \in (0, 1)$ such that, for every $\alpha \in (1-\epsilon, 1)$, the origin is a globally finite time stable equilibrium for the system (6) under the feedback control

$$u = -k_1|x_1|^{\alpha_1}\mathrm{sign}(x_1) - \cdots - k_n|x_n|^{\alpha_n}\mathrm{sign}(x_n), \tag{7}$$

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, \quad i = 2, \ldots, n,$$

with $\alpha_{n+1} = 1$ and $\alpha_n = \alpha$.

Consider the following uncertain second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + d, \end{aligned} \tag{8}$$

where $x = [x_1, x_2]^T$ is the state vector, $u = u_{\text{nominal}}$ is the control input and d is a matched disturbance. Also assuming $\Delta_0 \geq |d|$ and $\Delta_1 \geq |\dot{d}|$.

Bhat and Bernstein Control

$$u_{\text{nominal}} = -k_1|x_1|^{\frac{1}{3}}\operatorname{sign}(x_1) - k_2|x_2|^{\frac{1}{2}}\operatorname{sign}(x_2) \tag{9}$$

where $k_1 = 20$ and $k_2 = 9$, and disturbance is $8\sin(t) + 4$ selected for the simulation.

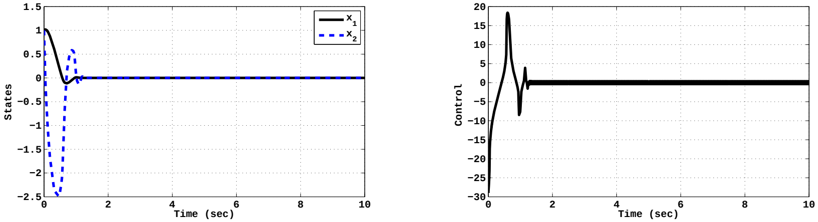


Figure: Finite Time control Bhat et.al., Without Disturbance

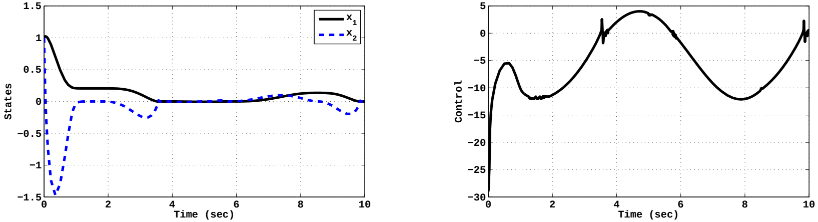


Figure: Finite Time control Bhat et.al., With Disturbance

Integral Sliding Mode Control with Discontinuous Control

- Integral sliding mode control has two parts viz. (i) *nominal control* (u_{nominal}) and (ii) a *discontinuous control* (u_{discon}).
- The nominal control is designed for the system without disturbance to have a desired trajectory. The design of u_{nominal} and u_{discon} are completely independent.
- It may be noted that the equivalent value of the discontinuous control is the negative of the disturbance.
- So, when $u = u_{\text{nominal}} + u_{\text{discon}}$ is applied to the system having disturbance, u_{discon} rejects the disturbance and the desired trajectory is obtained by the application of u_{nominal}.

ISM control input, $u = u_{\text{nominal}} + u_{\text{discon}}$

$$u_{\text{discon}} = -k_3 \text{sign}(s), \qquad k_3 > \Delta_0. \tag{10}$$

The sliding surface $s \in \mathbb{R}$ is defined as

$$s = x_2 - x_{20} - \int_0^t u_{\text{nominal}} \, d\tau, \tag{11}$$

Integral Sliding Mode Control with Discontinuous Control

- When the system is on the sliding surface, the equivalent value of the control is calculated by substituting the derivative of the sliding surface equal to zero.

Hence, mathematically one can write

$$\begin{aligned} \dot{s} &= u + d - u_{\text{nominal}} = 0 \\ &= u_{\text{nominal}} + u_{\text{discon}} + d - u_{\text{nominal}} = 0 \\ &\Rightarrow u_{\text{discon}} = -d \end{aligned} \tag{12}$$

- Substituting the value of u_{discon} from the (10), one can write that

$$[-k_3 \text{sign}(s)]_{eq} + d = 0 \tag{13}$$

- Therefore, when system is on the sliding surface, the value of the disturbance is $d = [k\text{sign}(s)]_{eq}$ and it is canceled out.
- But, it can clearly be observed here that the original control u, which is applied to the plant (8), also contains the discontinuous control.
- Due to the discontinuity in the control chattering phenomenon is generated in actuators, and it is not desirable for the practical implementation.

For the simulation, control gains are chosen as $k_1 = 20$, $k_2 = 9$, $k_3 = 20$ and disturbance as $8\sin(t) + 4$.

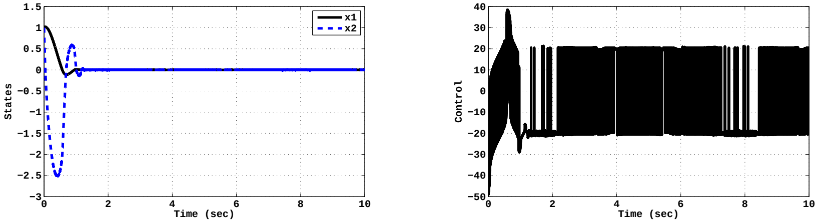


Figure: Discontinuous Integral Sliding Mode Control with disturbance.

Proposed Continuous Integral SMC with STC as a Disturbance Observer

- Consider again the same second order system (8), but now the control input for the system is given by $u = u_{\text{nominal}} + u_{\text{STC}}$, where u_{nominal} is nominal control chosen (9)

- u_{STC} is super twisting control given as

$$\begin{aligned} u_{\text{STC}} &= -k_4|s|^{\frac{1}{2}}\text{sign}(s) + v \\ \dot{v} &= -k_5\text{sign}(s), \end{aligned} \tag{14}$$

- Sliding surface s is defined as (11) and $k_4 = 1.5\sqrt{\Delta_1}$, $k_5 = 1.1\Delta_1$. Here also the sliding surface is so designed that sliding mode starts from the initial time.

Proposed Continuous Integral SMC with STC as a Disturbance Observer

- Differentiating (11) we get

$$\dot{s} = u + d - u_{\text{nominal}} \tag{15}$$

- Substituting the value of u one gets

$$\dot{s} = u_{\text{STC}} + d \tag{16}$$

- After substituting the value of control (14) to (16), one can write

$$\begin{aligned} \dot{s} &= -k_4 |s|^{\frac{1}{2}} \text{sign}(s) + v + d \\ \dot{v} &= -k_5 \text{sign}(s) \end{aligned} \tag{17}$$

- Let us define $z = v + d$, so $\dot{z} = \dot{v} + \dot{d}$. After substituting value of z in (17), one can write

$$\begin{aligned} \dot{s} &= -k_4 |s|^{\frac{1}{2}} \text{sign}(s) + z \\ \dot{z} &= -k_5 \text{sign}(s) + \dot{d} \end{aligned} \tag{18}$$

Proposed Continuous Integral SMC with STC as a Disturbance Observer

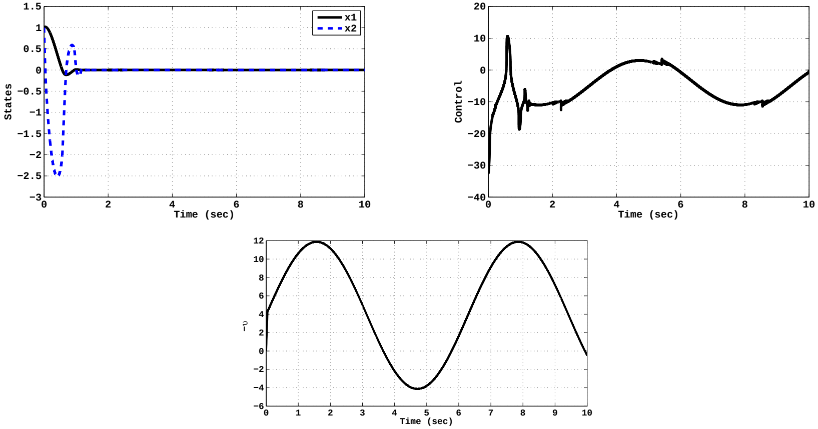


Figure: Continuous Integral Sliding Mode Control with disturbance.

Proposed Continuous Integral SMC with STC as a Disturbance Observer

Consider the following uncertain chain of integrators

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= u + d
\end{aligned}
\tag{19}
$$

where d is Lipschitz disturbance and $|\dot{d}| \leq d_0$ is bounded.

Proposed Continuous Integral SMC with STC as a Disturbance Observer

Theorem

Let $k_1, \ldots, k_n > 0$ be such that the polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$ is Hurwitz, and there exists $k_{n+1}, k_{n+2} > 0$, $\epsilon \in (0,1)$ such that, for every $\alpha \in (1-\epsilon, 1)$, the origin is a globally finite time stable equilibrium for the uncertain system (19) under the feedback control

$$u = u_{nominal} + u_{STC} \tag{20}$$

where

$$u_{nominal} = -k_1|x_1|^{\alpha_1} sign(x_1) - \cdots - k_n|x_n|^{\alpha_n} sign(x_n) \tag{21}$$

and

$$\begin{aligned} u_{STC} &= -k_{n+1}|s|^{\frac{1}{2}} sign(s) + v \\ \dot{v} &= -k_{n+2} sign(s), \end{aligned} \tag{22}$$

with the sliding surface $s \in \mathbb{R}$ is proposed as

$$s = x_n - x_{n0} - \int_0^t u_{nominal} d\tau \tag{23}$$

where x_{n0} is the initial value of the state variable x_n and $\alpha_1, \ldots, \alpha_n$ satisfy

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, \quad \forall i = 2, \ldots, n,$$

with $\alpha_{n+1} = 1$ and $\alpha_n = \alpha$.

Proposed Continuous Integral SMC with STC as a Disturbance Observer

- It is not always possible to convert any system in the form of the uncertain chain of integrators.

Generalized Continuous Integral Like SMC

Consider the system of the following form

$$\dot{x} = Ax + B(u + d) \tag{24}$$

where $x \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}$, $d \in \mathbb{R}$ are the state, system matrix, input matrix, control and disturbance respectively.

- Control input for the system (24) is given by $u = u_{\text{nominal}} + u_{\text{STC}}$, where u_{nominal} is nominal control and u_{STC} is super twisting control.
- The nominal control u_{nominal} is designed to achieve the specified performance, when system is free from disturbances.
- For example:-u_{nominal} can be any linear control, PID, LQR (linear quadratic regulator), state feedback, optimal control, time varying control, adaptive control etc.

Proposed Continuous Integral SMC with STC as a Disturbance Observer

Required sliding surface for the system (24) is defined as

$$s = G\left[x(t) - x(t_0) - \int_0^t (Ax + Bu_{\text{nominal}})d\tau\right] \tag{25}$$

where $G \in \mathbb{R}^{1\times n}$, $x(t_0)$ are the projection matrix and initial condition of the system respectively. Sliding surface is chosen such that the system trajectories start from the sliding surface and if disturbance comes into the picture then u_{STC} becomes active and disturbance becomes compensated. Mathematically, above procedure can be explained as

$$\begin{aligned}\dot{s} &= G\left[Ax + Bu + d - Ax - Bu_{\text{nominal}}\right]\\ &= G\left[Ax + B(u_{\text{nominal}} + u_{\text{STC}} + d) - Ax - Bu_{\text{nominal}}\right]\\ &= GB\left[u_{\text{STC}} + d\right]\end{aligned}$$

Without loss of generality let us consider $GB = 1$ (otherwise u_{STC} control is scaled by $(GB)^{-1}$, one can get

$$\dot{s} = u_{\text{STC}} + d$$

Further analysis is similar as previous discussions.

Application of Continuous ISMC for the Position Control of Industrial Emulator

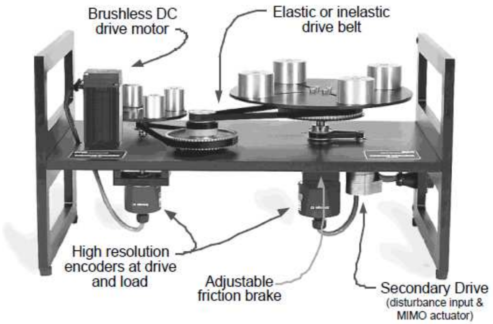


Figure: Industrial Emulator Setup

Application of Continuous ISMC for the Position Control of Industrial Emulator

Rigid Body Plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -8.4344 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 458.46 \end{bmatrix} (u + d)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{26}$$

where x_1, x_2 are the angular position and the angular velocity of the load disk, u is the input voltage to the drive motor and d is the disturbance voltage signal injected externally to perturb the plant. The purpose of giving an external disturbance is to check the robustness property of the system with the proposed control. The disturbance term d is taken to be $d = 0.4\sin(t)$.

Flexible Drive Plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -209.6 & -2 & 838.4 & 1.7 \\ 0 & 0 & 0 & 1 \\ 77.9 & 0.15 & -311.8 & -2.47 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2306 \\ 0 \\ 0 \end{bmatrix} (u + d)$$
$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{27}$$

where x_1, x_3 are the angular position of the drive disk and load disk and x_2, x_4 are the angular velocity of the drive disk and load disk respectively. Same disturbance is used in this configuration as well to test robustness of the proposed control.

LQR Controller as a Nominal Control

- The LQR controller design parameter for the rigid body plant is chosen as

$$Q_1 = \begin{bmatrix} 30 & 0 \\ 0 & 0.15 \end{bmatrix}, \quad R_1 = 10$$

and gain matrix

$$K_1 = \begin{bmatrix} 1.73 & 0.15 \end{bmatrix}.$$

- Similarly, design parameter for the flexible drive plant is chosen as

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_2 = 1$$

and gain matrix

$$K_2 = \begin{bmatrix} 0.2547 & 0.0142 & -0.018 & 0.046 \end{bmatrix}.$$

Proposed Controller

$$s = G\left[e(t) - e(t_0) - \int_0^t (Ae + Bu_{\text{nominal}})d\tau\right]$$

and the super twisting controller u_{STC} becomes

$$\begin{aligned} u_{\text{STC}} &= (GB)^{-1}(-k_4|s|^{\frac{1}{2}}\text{sign}(s) + z) \\ \dot{z} &= -k_5\text{sign}(s) \end{aligned} \tag{28}$$

- Rigid body plant configuration $k_4 = 1$, $k_5 = 0.45$ and $G = [1 \;\; 1/458.46]$.
- Flexible drive plant configuration $k_4 = 1$, $k_5 = 0.45$ and $G = [1 \;\; 1/2306 \;\; 1 \;\; 1]$.

Discontinuous ISMC

$$u_{\text{discon}} = (GB)^{-1}(-k_3\text{sign}(s))$$

Control gain $k_3 = 0.5$ is chosen for simulation and experiment in the both configuration.

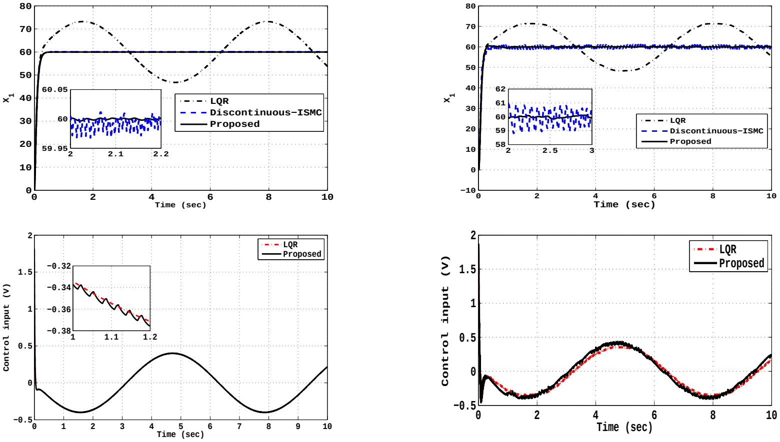


Figure: Simulation and Experimental Results: Rigid Body Plant

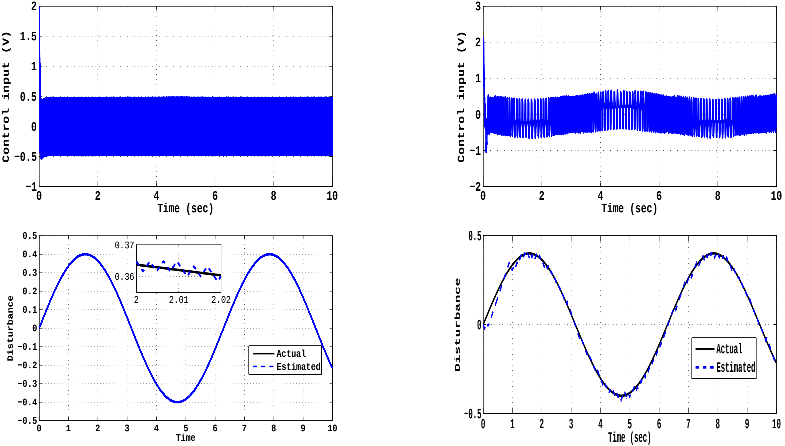


Figure: Simulation and Experimental Results: Rigid Body Plant

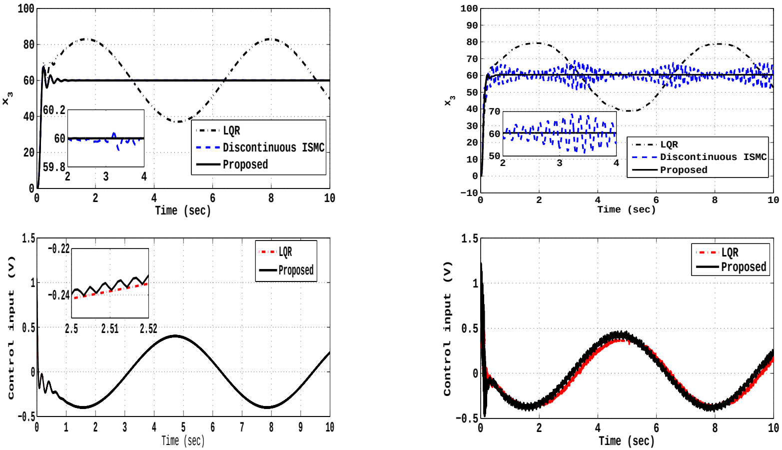
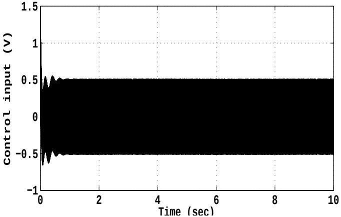


Figure: Simulation and Experimental Results: Flexible Drive Plant



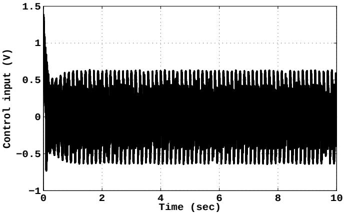


Figure: Simulation and Experimental Results: Flexible Drive Plant

Conclusions

- ISM control is able to reject the disturbance but it has discontinuous control.
- Continuous ISM proposed here overcomes the discontinuous nature of control by exploiting the super twisting algorithm.
- From the practical point of view the proposed controller is very much useful.

How to implement super-twisting controller based on sliding mode observer?

Consider the second order dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \rho_1 \\ y &= x_1 \end{aligned} \tag{29}$$

- y is the output of the system.
- ρ_1 is a matched disturbance/uncertainties.

- Most of the controller needs all the state information, so first we reconstruct the other state of the system.
- Then we design a super twisting controller based on the estimated information.

Super twisting observer dynamics for system (29)

$$\begin{aligned}\dot{\hat{x}}_1 &= k_1|e_1|^{\frac{1}{2}}\text{sign}(e_1) + \hat{x}_2 \\ \dot{\hat{x}}_2 &= k_2\text{sign}(e_1) + u\end{aligned} \tag{30}$$

Let us define the error $e_1 = x_1 - \hat{x}_1$ and $e_2 = x_2 - \hat{x}_2$.

STC based on STO

Now, the error dynamics is

$$\begin{aligned}\dot{e}_1 &= -k_1|e_1|^{\frac{1}{2}}\text{sign}(e_1) + e_2 \\ \dot{e}_2 &= -k_2\text{sign}(e_1) + \rho_1\end{aligned} \tag{31}$$

- It is assumed that $|\rho_1| < \Delta_0$ and Δ_0 is known.
- If we choose $k_1 = 1.5\sqrt{\Delta_0}$ and $k_2 = 1.1\Delta_0$ then error dynamics will goes to zero.
- Once the error e_1 and e_2 is zero, one can say that $x_1 = \hat{x}_1$ and $x_2 = \hat{x}_2$ after finite time $t > T_0$.

- System represented by (29) has relative degree two system with respect to output variable $y = x_1$.
- Therefore one cannot apply the direct STC, because it is applicable for only relative degree one system.
- So we have to define a sliding manifold of the following form to get a relative degree one with respect to sliding manifold.

$$\hat{s} = c_1 x_1 + \hat{x}_2 = 0, \quad \text{where } c_1 > 0 \tag{32}$$

STC based On STO

To synthesize the control law, taking the time derivative of (32)

$$\begin{aligned} \dot{\hat{s}} &= c_1 \dot{x}_1 + \dot{\hat{x}}_2 \\ \dot{\hat{s}} &= c_1 x_2 + u + k_2 \text{sign}(e_1) \end{aligned} \tag{33}$$

Substituting the $x_2 = e_2 + \hat{x}_2$ in the (33), then we can write as

$$\dot{\hat{s}} = c_1 \hat{x}_2 + c_1 e_2 + u + k_2 \text{sign}(e_1). \tag{34}$$

System (29) in the co-ordinate of x_1 and $\hat{s}$ by using (32) and (34),

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 + e_2 \\ \dot{\hat{s}} &= c_1 \hat{x}_2 + c_1 e_2 + u + k_2 \text{sign}(e_1).\end{aligned} \tag{35}$$

Now if we will select the control u as

$$u = -c_1 \hat{x}_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) - \int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau. \tag{36}$$

where λ_1 and λ_2 are the designed parameters for the controller.

Substituting the control input (36) in (35),

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 + e_2 \\ \dot{\hat{s}} &= c_1 e_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) - \int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau + k_2 \text{sign}(e_1).\end{aligned} \tag{37}$$

The overall closed loop system controller observer together

$$\Pi : \begin{cases} \dot{x}_1 & = \hat{s} - c_1 x_1 + e_2 \\ \dot{\hat{s}} & = c_1 e_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) - \int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau + k_2 \text{sign}(e_1) \end{cases}$$

$$\Xi : \begin{cases} \dot{e}_1 & = -k_1 |e_1|^{\frac{1}{2}} \text{sign}(e_1) + e_2 \\ \dot{e}_2 & = -k_2 \text{sign}(e_1) + \rho_1. \end{cases}$$

STC based On STO

- It is already discussed earlier that estimation error of system Ξ converges to zero in finite time, i.e. there exists a $T_0 > 0$ such that for all $t > T_0$, it follows that $e_1 = e_2 = 0$.
- Note that the trajectories of system Π cannot escape to infinity in finite time.
- Usually observer gains are chosen in such a way that observation error converges faster.

The closed loop system further we can write as

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 \\ \dot{\hat{s}} &= -\lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) - \int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau + k_2 \text{sign}(e_1),\end{aligned} \tag{38}$$

In another way by adding some new fictitious state variable L we can write

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 \\ \dot{\hat{s}} &= -\lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) + L + k_2 \text{sign}(e_1) \\ \dot{L} &= -\lambda_2 \text{sign}(\hat{s})\end{aligned} \tag{39}$$

- It is clear from the above mathematical transformation that,by selecting STO to estimate the state of the second order uncertain system (29)and following the standard way of the STC design as (36) by taking sliding manifold as (32), the second order sliding motion never starts in (39).
- Because $\dot{\hat{s}}$ contains the non-differentiable term $k_2\text{sign}(e_1)$, which prevents the possibility of lower two subsystem of (39) to act as the super twisting.
- Thus second order sliding motion (so that $\hat{s} = \dot{\hat{s}} = 0$ in finite time) never begins.

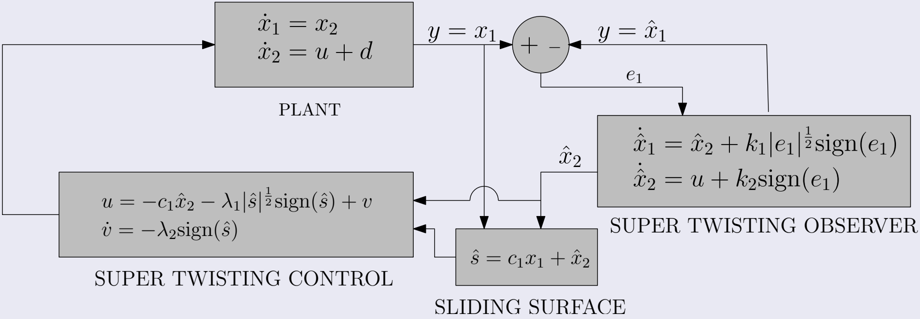


Figure: Block diagram of the Super Twisting Control based on Super Twisting Observer

Proposal 1

- The main aim here, is to design u, such that the second order sliding motion occurs in finite time.
- For this purpose control is selected as

$$u = -c_1\hat{x}_2 - k_2\text{sign}(e_1) - \lambda_1|\hat{s}|^{\frac{1}{2}}\text{sign}(\hat{s}) - \int_0^t \lambda_2\text{sign}(\hat{s})d\tau \tag{40}$$

where, $\lambda_1 > 0$ and $\lambda_2 > 0$ are controller parameter.

Proof

The closed loop system after substituting (40) into (35),

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 + e_2 \\ \dot{s} &= c_1 e_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) - \int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau \end{aligned} \tag{41}$$

- As discussed earlier, the observer error converges to zero in finite time so substituting $e_2 = 0$,

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 \\ \dot{\hat{s}} &= -\lambda_1 |\hat{s}|^{\frac{1}{2}} \text{sign}(\hat{s}) + \nu \\ \dot{\nu} &= -\lambda_2 \text{sign}(\hat{s}) \end{aligned} \tag{42}$$

where $\nu = -\int_0^t \lambda_2 \text{sign}(\hat{s}) d\tau$.

- The last two equations of (42) have the same structure as super twisting algorithm.
- Therefore, one can easily observe that after finite time $t > T_1$, $\hat{s} = \dot{\hat{s}} = 0$, which further implies, that the closed loop system is given as

$$\begin{aligned} \dot{x}_1 &= -c_1 x_1 \\ x_2 &= -c_1 x_1 \end{aligned} \tag{43}$$

- Therefore, both the states x_1 and x_2 are asymptotically stable by choosing $c_1 > 0$.

STC based On STO

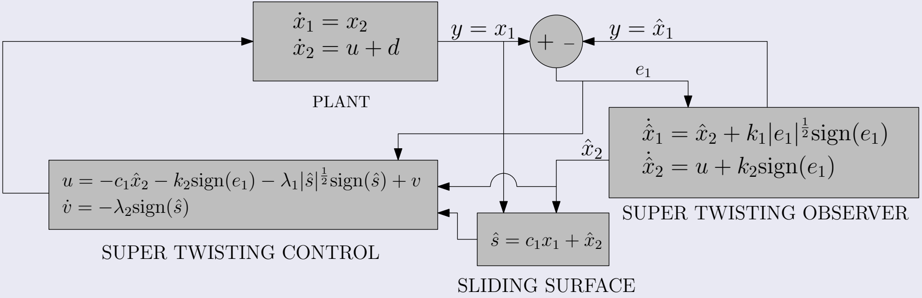


Figure: Block diagram of the Proposition 1 control based on Super Twisting Observer

Consider the following sliding sliding surface

$$s = c_1 x_1 + x_2 = 0 \tag{44}$$

Assuming that entire state vector is available.

For realizing the super twisting control expression

Take the first time derivative of sliding surface s (44),

$$\dot{s} = c_1 \dot{x}_1 + \dot{x}_2 \tag{45}$$

Now substitute $\dot{x}_1$ and $\dot{x}_2$ from (29) into (45), one can write

$$\dot{s} = c_1 x_2 + u + \rho_1 \tag{46}$$

Now design control as

$$u = -c_1 x_2 - \lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) d\tau \tag{47}$$

assuming that both the states are available for measurement.

After substituting the control (47) into (46), one can write

$$\dot{s} = -\lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) - \int_0^t \lambda_2 \text{sign}(s) d\tau + \rho_1, \tag{48}$$

or

$$\begin{aligned} \dot{s} &= -\lambda_1 |s|^{\frac{1}{2}} \text{sign}(s) + z \\ \dot{z} &= -\lambda_2 \text{sign}(s) + \dot{\rho}_1. \end{aligned} \tag{49}$$

where $z = \nu_1 + \rho_1$ and $\nu_1 = -\int_0^t \lambda_2 \text{sign}(s) d\tau$.

- It is assumed that $|\dot{\rho}_1| < \Delta_1$.
- Now select $\lambda_1 = 1.5\sqrt{\Delta_1}$ and $\lambda_2 = 1.1\Delta_1$, which leads to second order sliding in finite time provided ρ_1 is Lipschitz and $|\dot{\rho}_1| < \Delta_1$.
- When $s = 0$, then $x_1 = x_2 = 0$ asymptotically same as discussed earlier by selecting $c_1 > 0$.
- The control (47) is based on full state information, so we cannot implement it directly on system (29) because we do not have the information of x_2.

- If we use STO to estimate the $\hat{x}_2$ and using it in control (47) by replacing x_2 with its estimated value $\hat{x}_2$ then control input (47) becomes,

$$u = -c_1\hat{x}_2 - \lambda_1|\hat{s}|^{\frac{1}{2}}\text{sign}(\hat{s}) - \int_0^t \lambda_2\text{sign}(\hat{s})d\tau \tag{50}$$

where $\hat{s} = c_1x_1 + \hat{x}_2$.

- If the controller (50) is applied to (29) then it is not possible to get SOSM on the chosen surface.
- It is already discussed in earlier section that control input (36) which is same as (50) is not able to achieve SOSM on the sliding surface $\hat{s}$.
- Once the control is applied to the system, the system becomes (39) where discontinuous term is presents in the first derivative of the sliding surface which prevents the second order sliding mode on the chosen surface.
- So the method is not mathematically correct to get second order sliding mode.

- If we use this method for practical implementation it may work (which may not be true for all system), because most of the time controller implemented digitally through computer.
- It mean that controller is implemented at some fix sampling time, so the value of discontinuous term $k_2\text{sign}(e_1)$ will be constant during sampling interval.
- In STOF control approach one has to choose STO gains k_1 and k_2 based on the upper bound of the disturbance and STC gains λ_1 and λ_2 based on the upper bound of the derivative of the disturbance.

- It is seen that if an n^{th} order SMO is used to estimate the states of n^{th} order perturbed integrator system, either the SOSM is not achieved or the controller becomes discontinuous.
- So a continuous controller design is proposed based on $(n+1)^{th}$ order observer.

The higher order sliding mode observer (HOSMO) dynamics for (29)

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + k_1|e_1|^{\frac{2}{3}}\text{sign}(e_1)\\
\dot{\hat{x}}_2 &= \hat{x}_3 + u + k_2|e_1|^{\frac{1}{3}}\text{sign}(e_1)\\
\dot{\hat{x}}_3 &= k_3\text{sign}(e_1) \qquad (51)
\end{aligned}$$

Let us define the error as $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$.

If we consider the perturbed n^{th} order integrator system,

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \qquad i = 1, \cdots, n-1 \\ \dot{x}_n &= u + \rho_1 \\ y &= x_1 \end{aligned} \tag{52}$$

the $(n+1)^{th}$ order SMO is given as

HOSMO for perturbed n^{th} order integrator

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + z_i \qquad i = 1, \cdots, n-1 \\ \dot{\hat{x}}_n &= \hat{x}_{n+1} + u + z_n \\ \dot{\hat{x}}_{n+1} &= z_{n+1} \end{aligned} \tag{53}$$

where the correction terms are given as

$$\begin{aligned} z_i &= k_i |e_1|^{(n-i+1)/(n+1)} \text{sign}(e_1) \qquad i = 1, \cdots, n-1 \\ z_{n+1} &= k_{n+1} \text{sign}(e_1) \end{aligned}$$

HOSMO based STC

Defining the error variables as $e_i = x_i - \hat{x}_i$ for $i = 1, \cdots, n$, we have

Error dynamics

$$\begin{aligned}
\dot{e}_1 &= -k_1|e_1|^{n/(n+1)}\text{sign}(e_1) + e_2 \\
\dot{e}_2 &= -k_2|e_1|^{(n-1)/(n+1)}\text{sign}(e_1) + e_3 \\
&\vdots \\
\dot{e}_n &= -k_n|e_1|^{1/(n+1)}\text{sign}(e_1) - \hat{x}_{n+1} + \rho_1 \\
\dot{\hat{x}}_{n+1} &= k_{n+1}\text{sign}(e_1)
\end{aligned} \tag{54}$$

Now define the new variable $e_{n+1} = -\hat{x}_{n+1} + \rho_1$. Also assuming ρ_1 is Lipschitz and $|\dot{\rho}_1| < \Delta_1$.

$$\begin{aligned}
\dot{e}_1 &= -k_1|e_1|^{n/(n+1)}\text{sign}(e_1) + e_2 \\
\dot{e}_2 &= -k_2|e_1|^{(n-1)/(n+1)}\text{sign}(e_1) + e_3 \\
&\vdots \\
\dot{e}_n &= -k_n|e_1|^{1/(n+1)}\text{sign}(e_1) - \hat{x}_{n+1} + e_{n+1} \\
\dot{e}_{n+1} &= -k_{n+1}\text{sign}(e_1) + \dot{\rho}_1
\end{aligned} \tag{55}$$

- The error dynamics in (55) has the structure of n^{th} order differentiator.
- So the error variables will converge to zero in finite time $t > T_0$ by appropriate choice of gains k_i.
- The value of k_{n+1} is dependent on the first derivative of ρ_1.

The error dynamics for (51) can be written as

$$\begin{aligned} \dot{e}_1 &= -k_1|e_1|^{\frac{2}{3}}\text{sign}(e_1) + e_2 \\ \dot{e}_2 &= -k_2|e_1|^{\frac{1}{3}}\text{sign}(e_1) - \hat{x}_3 + \rho_1 \\ \dot{\hat{x}}_3 &= k_3\text{sign}(e_1) \end{aligned} \tag{56}$$

Now define the new variable $e_3 = -\hat{x}_3 + \rho_1$. Also assuming ρ_1 is Lipschitz and $|\dot{\rho}_1| < \Delta_1$.

HOSMO based STC

$$\begin{aligned}
\dot{e}_1 &= -k_1|e_1|^{\frac{2}{3}}\text{sign}(e_1) + e_2 \\
\dot{e}_2 &= -k_2|e_1|^{\frac{1}{3}}\text{sign}(e_1) + e_3 \\
\dot{e}_3 &= -k_3\text{sign}(e_1) + \dot{\rho}_1
\end{aligned} \tag{57}$$

- The above equation is finite time stable which is already proved in literature.
- So we can conclude that e_1, e_2 and e_3 will converge to zero in finite time $t > T_2$, by selecting the appropriate gains k_1, k_2 and k_3.
- One can find $x_1 = \hat{x}_1$, $x_2 = \hat{x}_2$ and $\hat{x}_3 = \rho_1$ after finite time $t > T_2$.

To design a super twisting control for a system (29) considering the same sliding surface (32) and taking its time derivative then, one can write expression as

$$\dot{\hat{s}} = c_1\dot{x}_1 + \dot{\hat{x}}_2$$
$$\dot{\hat{s}} = c_1\hat{x}_2 + c_1e_2 + u + k_2|e_1|^{\frac{1}{3}}\text{sign}(e_1) + \int_0^t k_3\text{sign}(e_1)d\tau \tag{58}$$

Representing the system (29) in the co-ordinate of x_1 and $\hat{s}$ by using (32) and (58),

$$\begin{aligned}
\dot{x}_1 &= \hat{s} - c_1 x_1 + e_2 \\
\dot{\hat{s}} &= c_1 \hat{x}_2 + c_1 e_2 + u + k_2 |e_1|^{\frac{1}{3}} \text{sign}(e_1) + \int_0^t k_3 \text{sign}(e_1) d\tau
\end{aligned} \tag{59}$$

Now, the control input u is selected as

$$u = -c_1\hat{x}_2 - k_2|e_1|^{\frac{1}{3}}\text{sign}(e_1) - \int_0^t k_3\text{sign}(e_1)d\tau - \lambda_1|\hat{s}|^{\frac{1}{2}}\text{sign}(\hat{s}) - \int_0^t \lambda_2\text{sign}(\hat{s})d\tau \tag{60}$$

or

$$u = -c_1\hat{x}_2 - \int_0^t k_3\text{sign}(e_1)d\tau - \lambda_1|\hat{s}|^{\frac{1}{2}}\text{sign}(\hat{s}) - \int_0^t \lambda_2\text{sign}(\hat{s})d\tau \tag{61}$$

Substituting the control input (60) in the (59),

$$
\begin{aligned}
\dot{x}_1 &= \hat{s} - c_1 x_1 + e_2 \\
\dot{\hat{s}} &= c_1 e_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \mathrm{sign}(\hat{s}) + v \\
\dot{v} &= -\lambda_2 \mathrm{sign}(\hat{s})
\end{aligned}
\tag{62}
$$

Now, the overall closed loop system can be represented as

$$
\begin{aligned}
\Pi_1 : &\begin{cases} \dot{x}_1 &= \hat{s} - c_1 x_1 + e_2 \\ \dot{\hat{s}} &= c_1 e_2 - \lambda_1 |\hat{s}|^{\frac{1}{2}} \operatorname{sign}(\hat{s}) + v \\ \dot{v} &= -\lambda_2 \operatorname{sign}(\hat{s}), \end{cases} \\
\Xi_1 : &\begin{cases} \dot{e}_1 &= -k_1 |e_1|^{\frac{2}{3}} \operatorname{sign}(e_1) + e_2 \\ \dot{e}_2 &= -k_2 |e_1|^{\frac{1}{3}} \operatorname{sign}(e_1) + e_3 \\ \dot{e}_3 &= -k_3 \operatorname{sign}(e_1) + \dot{\rho}_1, \end{cases}
\end{aligned}
\tag{63}
$$

It is already discussed earlier that estimation error of system Ξ_1 converges to zero in finite time.
Note that the trajectories of system Π_1 (above) cannot escape to infinity in finite time.
So, we can substitute $e_1 = e_2 = 0$.

HOSMO based STC

Once the error becomes zero, the closed loop system is given by the following expression

$$\begin{aligned}\dot{x}_1 &= \hat{s} - c_1 x_1 \\ \dot{\hat{s}} &= -\lambda_1 |\hat{s}|^{\frac{1}{2}} \mathrm{sign}(\hat{s}) + v \\ \dot{v} &= -\lambda_2 \mathrm{sign}(\hat{s}) \end{aligned} \tag{64}$$

The lower two equation of (64) is super twisting equation, by selecting appropriate gains λ_1 and λ_2, then $\hat{s} = \dot{\hat{s}} = 0$ in finite time. which further implies, that the closed loop system is given as

$$
\begin{aligned}
\dot{x}_1 &= -c_1 x_1 \\
x_2 &= -c_1 x_1
\end{aligned}
\tag{65}
$$

Therefore, both the states x_1 and x_2 are asymptotically stable by choosing $c_1 > 0$.

HOSMO based STC

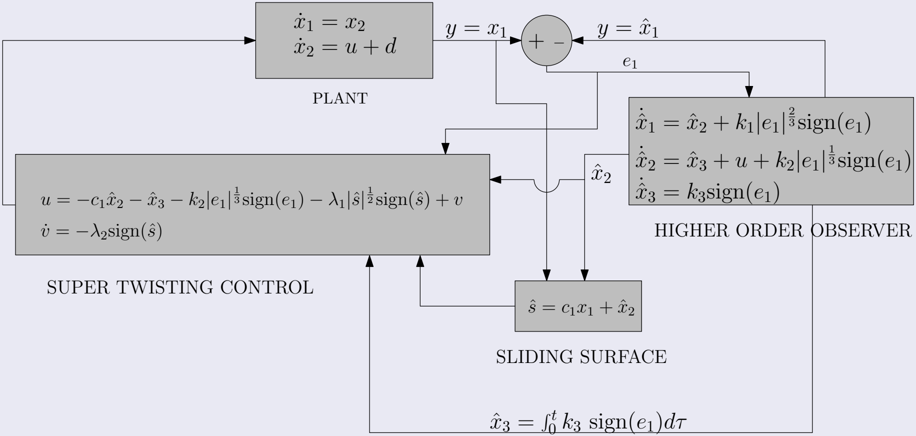


Figure: Block diagram of the Super Twisting Control based on HOSM Observer

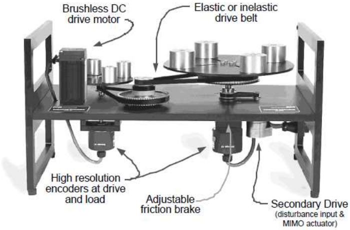
- It is clear from the STC control (60) and (61) expression based on HOSMO (51) is continuous.
- Also, when we design STC control based on HOSMO then one has to tune only the observer gains, according to the first derivative of disturbance, because it is necessary for the convergence of the error variables of the HOSMO.
- However, during controller design there is no explicit gain condition for the λ_2 with respect to the disturbances.

HOSMO based STC

- One can also observe that super twisting output feedback controller (50) design based on super twisting observer, requires more conditions on gain relating to disturbance and its derivative.
- One is STO gain means observer gain k_2 based on the explicit maximum bound of the direct disturbance.
- Another is λ_2, STC gain based on the maximum bound of the derivative of disturbance.

HOSMO based STC

- Therefore, one can conclude from the above observation that sound mathematical analysis reduces the two gains conditions with respect to disturbance by simply one gain condition.
- Also the precision of the sliding manifold is much improved by using the HOSMO based STC rather than STO based STC.
- Due to the increase of this precision of sliding variable precision of the states are also much affected.
- In other word if we talk about stabilization problem, then states are much closer to the origin in the case of HOSMO based STC rather than STO based STC.
- We only talk about the closeness of states variable with respect to an equilibrium point, because only asymptotic stability is possible in both the design methodology.



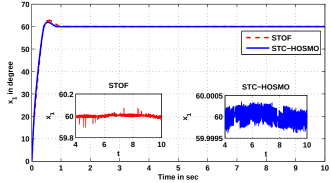
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -8.4344 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 458.46 \end{bmatrix} (u + d)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{66}$$

- x_1, x_2 are the angular position and the angular velocity of the load disk,
- u is the input voltage to the drive motor
- d is the disturbance voltage signal injected to perturb the plant, disturbance value is $d = 0.2\sin(t)$ considered.

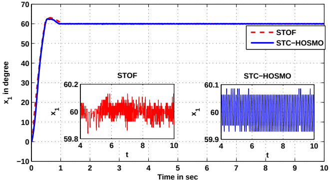
The controller and observer gains are selected as follows

- STC-STOF
  - STC gains $\lambda_1 = 2$ and $\lambda_2 = 2$
  - STO gains $k_1 = 1.5\sqrt{m}$ and $k_2 = 1.1m$, where $m = 100$.
- STC-HOSMO
  - STC gains $\lambda_1 = 2$ and $\lambda_2 = 2$
  - HOSMO gains $k_1 = 6n^{\frac{1}{3}}$, $k_2 = 11n^{\frac{1}{2}}$ and $k_3 = 6n$, where $n = 50$.

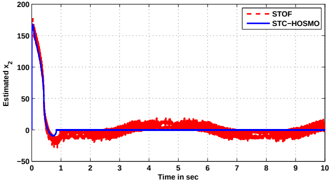
The sliding surface is chosen as $s = x_1 + \frac{1}{458.46}\hat{x}_2$.



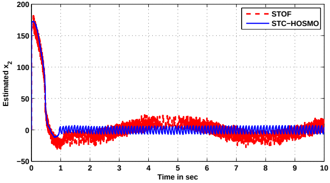
(a) Evolution of position (simulation)



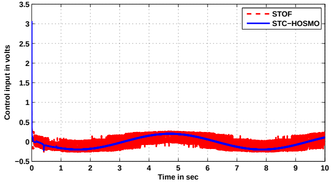
(b) Evolution of position (experimental)



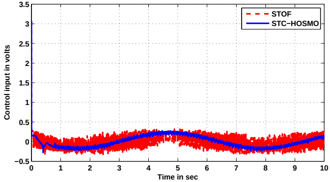
(c) Evolution of estimated state x_2 (simulation)



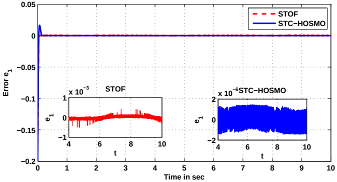
(d) Evolution of estimated state x_2 (experimental)



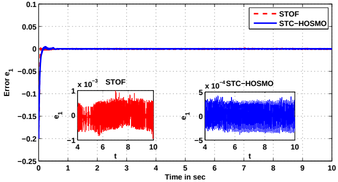
(e) Evolution of control input u (simulation)



(f) Evolution of control input u (experimental)



(g) Evolution of observer error e_1 (simulation)



(h) Evolution of observer error e_1 (experimental)

Conclusion

- It is shown that, if one wants to implement absolutely continuous STC signal for the perturbed double integrator, the derivative of the chosen switching function must be Lipschitz in the time.
- Therefore, we have the need of third order observer in this case.
- The same is also true for the higher order perturbed chain of integrators, when we want to synthesize absolutely continuous STC signal under the output information.
- Experimental results are also presented to support the effectiveness of the proposed methodology.

Thank You!