IEEE IES Distinguished Lecture

Prof. B. Bandyopadhyay

FNAE, IEEE Fellow, Institute Chair Professor and Head
Systems and Control Engineering Group
Indian Institute of Technology Bombay
India

International Summer School on Sliding Mode Control- Variable Structure Systems
State University of Rio de Janeiro (UERJ), Rio de Janeiro, Brazil
April 8-12, 2019

Event-Triggered Sliding Mode Control

Outline

Outline

Introduction

Mainly two ways of control implementation:

- Periodic
- Aperiodic

Periodic Implementation (Fixed Sampling Period)

- Easy to implement
- Simple techniques are available to analyse the system stability e.g., discrete-time approach, emulation approach, etc.

Aperiodic Implementation (Varying Sampling Period)

- Minimize the resource utilization
- Analyzing the system stability is a challenging task

Mainly two ways of control implementation:

- Periodic
- Aperiodic

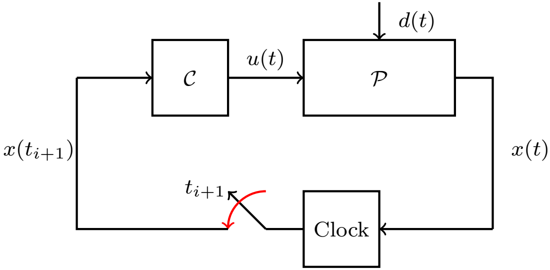
Periodic Implementation (Fixed Sampling Period)

- Easy to implement
- Simple techniques are available to analyse the system stability e.g., discrete-time approach, emulation approach, etc.

Aperiodic Implementation (Varying Sampling Period)

- Minimize the resource utilization
- Analyzing the system stability is a challenging task

Event-triggering is one of the aperiodic control implementation strategies that guarantees the system stability.



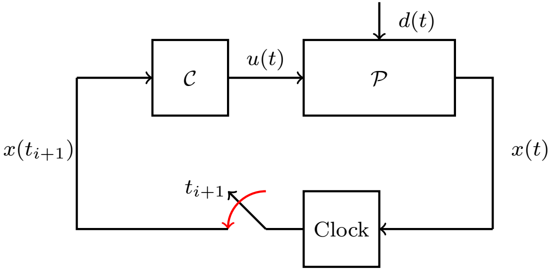
Sampled-Data System

- The periodic control update

$$t_{i+1} = t_i + h$$

- Inefficient in terms of resource use
- Open-loop sampling, i,e., clock is reset once the time period h is elapsed irrespective of the state evolution

Introduction: Why Event-Triggering?



Sampled-Data System

- The periodic control update

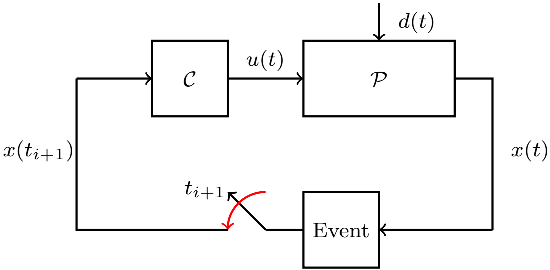
$$t_{i+1} = t_i + h$$

- Inefficient in terms of resource use
- Open-loop sampling, i,e., clock is reset once the time period h is elapsed irrespective of the state evolution

Demerits

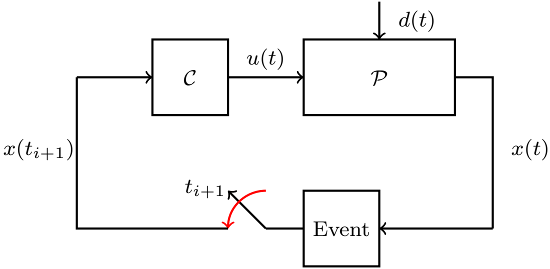
- More control updates even when there is no demand
- Not suitable for parallel tasks
- Inter sample behaviour may be difficult to analyse

Introduction: What is Event-Triggering?



Event-Triggering Strategy

- Control is updated in aperiodic manner
- Triggering rule (event) governs the control updates
- Event generator uses the system state trajectories, so it is closed-loop sampling



Event-Triggering Strategy

- Control is updated in aperiodic manner
- Triggering rule (event) governs the control updates
- Event generator uses the system state trajectories, so it is closed-loop sampling

Advantages

- A need based approach:- so resources are used optimally
- Highly desired in networked control system

Preliminaries: Event-Triggering

Consider

$$\dot{x} = f(x, u), \qquad x_0 = x(t_0)$$

where $u = \kappa(x)$ is any given stabilizing control law such that closed loop system

$$\dot{x} = f(x, \kappa(x))$$

is asymptotically stable. Assume control is implemented as

$$u(t) = u(t_i) = \kappa(x(t_i)) \quad \forall t \in [t_i, t_{i+1}).$$

Define the error $e(t) := x(t_i) - x(t)$. There exists a Lyapunov function V such that

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(\|x\|) + \gamma(\|e\|)$$

where $\alpha(\cdot)$ and $\gamma(\cdot)$ are $\mathcal{K}_\infty$ functions.[a]

[a] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class-$\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha(\cdot)$ is said to be class-$\mathcal{K}_\infty$ if it belongs to class-$\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$.

Introduction: What is Event-Triggering?

Let there exists compact sets for $\alpha(\cdot)$ and $\gamma(\cdot)$. Assuming $\alpha^{-1}(\cdot)$ and $\gamma(\cdot)$ are Lipschitz on these compacts, the triggering rule based on Lyapunov method is for some $\sigma \in (0,1)$ (Tabuada'2006)

$$\gamma(\|e\|) \leq \sigma\alpha(\|x\|) \quad \Longleftrightarrow \quad \alpha^{-1}\left(\frac{\gamma(\|e\|)}{\sigma}\right) \leq \|x\|.$$

Since $\alpha^{-1}(\cdot)$ and $\gamma(\cdot)$ are Lipschitz, we write the following for a constant P

$$\alpha^{-1}\left(\frac{\gamma(\|e\|)}{\sigma}\right) \leq \frac{P}{\sigma}\|e\| \leq \|x\|.$$

Triggering Schemes

$$t_{i+1} = \inf\left\{t > t_i : \|e(t)\| > \frac{\sigma}{P}\|x(t)\|\right\}$$

This triggering rule always ensures $\gamma(\|e\|) \leq \sigma\alpha(\|x\|)$. Thus, we have

$$\begin{aligned}\dot{V} &= -\alpha(\|x\|) + \gamma(\|e\|)\\ &\leq -(1-\sigma)\alpha(\|x\|) < 0.\end{aligned}$$

Outline

Event-Triggered Sliding Mode

Why Sliding Mode Control?

- Gives robust performance in the presence of matched disturbances
- Sliding mode dynamics becomes disturbance free system

Why Event-Triggered SMC?

- To achieve robust performance with event-triggering strategy
- To achieve improved steady state performance with discrete implementation strategy

Sliding Mode Control

Consider a LTI System:

$$\dot{x} = Ax + Bu + Bd$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. Define the sliding variable $s = c^\top x$ and sliding manifold as

$$\mathcal{S} := \left\{ x \in \mathbb{R}^n : c^\top x = 0 \right\}.$$

It is well known that the control

$$u = (c^\top B)^{-1} \left(c^\top Ax + K \text{sign} s \right)$$

brings s to the sliding manifold $\mathcal{S}$ in finite time if $K \geq \sup_{t \geq 0} |c^\top Bd(t)| + \eta$ for some $\eta > 0$.

SMC With Event-Triggering

Sliding Mode Control

Consider a LTI System:

$$\dot{x} = Ax + Bu + Bd$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. Define the sliding variable $s = c^\top x$ and sliding manifold as

$$\mathcal{S} := \left\{ x \in \mathbb{R}^n : c^\top x = 0 \right\}.$$

It is well known that the control

$$u = (c^\top B)^{-1} \left(c^\top Ax + K \text{sign} s \right)$$

brings s to the sliding manifold $\mathcal{S}$ in finite time if $K \geq \sup_{t \geq 0} |c^\top Bd(t)| + \eta$ for some $\eta > 0$.

Implementation of Control

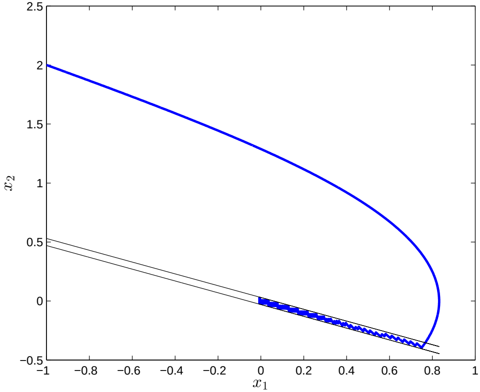
- Digital implementation never result $s = 0$ for all time
- Periodic control implementation results in a steady-state that depends on sampling period and also on disturbance bound
- Can there be a method to bound steady-state value in digital implementation?
  Yes Event-Triggering strategy

Definition: Practical Sliding Mode

Given $\Delta_1 > 0$ there exists a $t_1 \in [t_0, \infty)$ such that the system trajectories reach the region bounded by Δ_1 in the vicinity of the sliding manifold in time t_1 and remain there for all time $t \geq t_1$.

Definition: Practical Sliding Mode

Given $\Delta_1 > 0$ there exists a $t_1 \in [t_0, \infty)$ such that the system trajectories reach the region bounded by Δ_1 in the vicinity of the sliding manifold in time t_1 and remain there for all time $t \geq t_1$.



SMC With Event-Triggering

Discrete Control

The SMC with discrete implementation is

$$u(t) = -(c^\top B)^{-1} \left(c^\top A x(t_i) + K \text{sign} s(t_i) \right)$$

for all $t \in [t_i, t_{i+1})$. Here $s(t_i) = c^\top x(t_i)$. For sampled-data implementation

$$0 < h = t_{i+1} - t_i, \quad \forall i \in \mathbb{Z}_{\geq 0}.$$

In event-triggered implementation, the inter event time, $T_i = t_{i+1} - t_i$, is aperiodic in nature. So, for stability of the system

- Design a stabilizing triggering rule
- Show $T_i > \tau$ for some $\tau > 0$ and all $i \in \mathbb{Z}_{\geq 0}$

Theorem

Let $\alpha \in (0, \infty)$ such that

$$\left| c^\top A e(t) \right| < \alpha.$$

Then, the event-triggered SMC guarantees practical sliding mode in the system if

$$K > \sup_{t \geq 0} \left| c^\top B d(t) \right| + \eta + \alpha$$

for some $\eta > 0$.

Proof

Choose $V(s) = \frac{1}{2}s^2$. Consider $t \in [t_i, t_{i+1})$ and $i \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}\dot{V}(s(t)) &= s(t) \left(c^\top A x(t) - c^\top A x(t_i) - K \text{sign} s(t_i) + c^\top B d(t) \right) \\ &= -s(t) \left(c^\top A e(t) + K \text{sign} s(t_i) - c^\top B d(t) \right) \\ &\leq |s(t)| \left| c^\top A e(t) \right| - s(t) K \text{sign} s(t_i) + |s(t)| \left| c^\top B \right| d_{\max}.\end{aligned}$$

SMC With Event-Triggering

Proof

If $\text{sign}(s(t_i)) = \text{sign}(s(t))$, the second term equals $K|s(t)|$. Then,

$$\begin{aligned}\dot{V}(s(t)) &\leq -|s(t)| \left(K - \alpha - \left| c^\top B \right| d_{\max} \right) \\ &< -\eta |s(t)|.\end{aligned}$$

- This shows that the trajectories are attracted towards the sliding manifold and it reaches the manifold in finite-time.

SMC With Event-Triggering

Proof

If $\text{sign}(s(t_i)) = \text{sign}(s(t))$, the second term equals $K|s(t)|$. Then,

$$\begin{aligned}\dot{V}(s(t)) &\leq -|s(t)|\left(K - \alpha - \left|c^\top B\right| d_{\max}\right)\\ &< -\eta|s(t)|.\end{aligned}$$

- This shows that the trajectories are attracted towards the sliding manifold and it reaches the manifold in finite-time.
- Once the trajectory reaches the sliding manifold, $\text{sign}(s(t))$ does not remain same in some triggering interval $[t_i, t_{i+1})$, i.e., $\text{sign}(s(t_i)) \neq \text{sign}(s(t))$.

Proof

If $\text{sign}(s(t_i)) = \text{sign}(s(t))$, the second term equals $K|s(t)|$. Then,

$$\dot{V}(s(t)) \leq -|s(t)| \left(K - \alpha - \left| c^\top B \right| d_{\max} \right) \\ < -\eta |s(t)|.$$

- This shows that the trajectories are attracted towards the sliding manifold and it reaches the manifold in finite-time.
- Once the trajectory reaches the sliding manifold, $\text{sign}(s(t))$ does not remain same in some triggering interval $[t_i, t_{i+1})$, i.e., $\text{sign}(s(t_i)) \neq \text{sign}(s(t))$.
- However, the trajectory remains bounded in the vicinity of sliding manifold. Now, using the relation $\|c\|\|A\|\|e(t)\| < \alpha$, we obtain

$$\begin{aligned} |s(t_i) - s(t)| &= \left| c^\top x(t_i) - c^\top x(t) \right| \\ &\leq \|c\|\|x(t_i) - x(t)\| \\ &= \|c\|\|e(t)\| \\ &< \frac{\alpha}{\|A\|}. \end{aligned}$$

- Thus, the trajectory remains within the region

$$\left\{ x \in \mathbb{R}^n : \left| c^\top x \right| < \frac{\alpha}{\|A\|} \right\}$$

once it enters this band.

This ensures the practical sliding mode in the system and the proof is completed.

State Evolution

Represent the system in regular form

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u + B_2d.$$

The sliding surface parameter is $c^\top = \begin{bmatrix} c_1^\top & 1 \end{bmatrix}$. Select c_1 such that $A_{11} - A_{12}c_1^\top$ is Hurwitz. Now, during sliding mode

$$\dot{x}_1 = (A_{11} - A_{12}c_1^\top)x_1 + A_{12}s$$
$$x_2 = -c_1^\top x_1 + s.$$

State Evolution

Represent the system in regular form

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u + B_2d.$$

The sliding surface parameter is $c^\top = \begin{bmatrix} c_1^\top & 1 \end{bmatrix}$. Select c_1 such that $A_{11} - A_{12}c_1^\top$ is Hurwitz. Now, during sliding mode

$$\dot{x}_1 = (A_{11} - A_{12}c_1^\top)x_1 + A_{12}s$$
$$x_2 = -c_1^\top x_1 + s.$$

Proposition

The system trajectories remain ultimately bounded during practical sliding mode within the region given by

$$\mathcal{B} = \left\{ x_1 \in \mathbb{R}^{n-1} : \|x_1\| \leq 2\alpha \frac{\|A_{12}^\top P\|}{\lambda_{\min}\{Q\}\|A\|} \right\}.$$

proof

Consider the Lyapunov function $V_1 = x_1^\top P x_1$. Then

$$\begin{aligned}\dot{V}_1 &= \dot{x}_1^\top P x_1 + x_1^\top P \dot{x}_1 \\ &\leq x_1^\top (A_{\text{cl}}^\top P + P A_{\text{cl}}) x_1 + 2\alpha \|A\|^{-1} A_{12}^\top P x_1\end{aligned}$$

where $A_{\text{cl}} = (A_{11} - A_{12} c_1^\top)$ is Hurtiwz. Then, for any $Q > 0$ such that $A_{\text{cl}}^\top P + P A_{\text{cl}} + Q = 0$. Using this,

$$\begin{aligned}\dot{V}_1 &\leq -x_1^\top Q x_1 + 2\alpha \|A\|^{-1} A_{12}^\top P x_1 \\ &\leq -\lambda_{\min}\{Q\} \|x_1\|^2 + 2\alpha \|A\|^{-1} \left\| A_{12}^\top P \right\| \|x_1\| \\ &= -\lambda_{\min}\{Q\} \left(\|x_1\| - 2\alpha \|A\|^{-1} \frac{\|A_{12}^\top P\|}{\lambda_{\min}\{Q\}} \right) \|x_1\|.\end{aligned}$$

We see that $\dot{V}_1 < 0$ whenever x_1 is outside the ball $\mathcal{B}$, and thus, $\mathcal{B}$ is attractive. This ensures that the system trajectories enter $\mathcal{B}$ in some finite time and remain ultimately bounded.

SMC With Event-Triggering

Design of Triggering Rule

Recall that the practical sliding mode is ensured if

$$\|c\|\|A\|\|e\| < \sigma\alpha \quad \sigma \in (0,1)$$

So, the triggering rule is proposed as

$$t_{i+1} = \inf\{t > t_i \ : \ \|c\|\|A\|\|e(t)\| \geq \sigma\alpha\}$$

Inter Event Time

Since triggering instants is generated whenever triggering rule is violated. So, we must ensure

- It is important to show

$$T_i := t_{i+1} - t_i > \tau \qquad \forall i \in \mathbb{Z}_{\geq 0}$$

for some $\tau > 0$.

Theorem

Let $\{t_i\}_{i=0}^{\infty}$ be triggering sequences. Then

$$t_{i+1} - t_i =: T_i \geq \frac{1}{\|A\|} \ln\left(1 + \sigma \frac{\alpha}{\|c\|(\rho(\|x(t_i)\|) + \beta)}\right)$$

where

$$\beta := \|B(c^\top B)^{-1}\|K + \|B\| d_{\max},$$

and

$$\rho(\|x(t_i)\|) := \left\|\left(A - B(c^\top B)^{-1} c^\top A\right) x(t_i)\right\|.$$

Proof

Construct $\Gamma = \{t : \|c\|\|A\|\|e(t)\| = 0\}$. Then for all time $t \in [t_i, t_{i+1}[\setminus\Gamma$, we write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\| &\leq \left\|\frac{\mathrm{d}}{\mathrm{d}t} e(t)\right\| = \left\|\frac{\mathrm{d}}{\mathrm{d}t} x(t)\right\| \\ &= \left\|Ax(t) - B(c^\top B)^{-1} c^\top Ax(t_i) - B(c^\top B)^{-1} K \mathrm{sign} s(t_i) + Bd(t)\right\|.\end{aligned}$$

Proof

Recall $x(t) = x(t_i) - e(t)$. Then, the above can be reduced to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\| &= \left\|Ax(t_i) - Ae(t) - B(c^\top B)^{-1}c^\top Ax(t_i) - B(c^\top B)^{-1}K\mathrm{sign}s(t_i) + Bd(t)\right\| \\ &\leq \|A\|\|e(t)\| + \left\|\left(A - B(c^\top B)^{-1}c^\top A\right)x(t_i)\right\| + \|B(c^\top B)^{-1}K\| + \|B\|d_{\max} \\ &= \|A\|\,\|e(t)\| + \rho(\|x(t_i)\|) + \beta.\end{aligned}$$

The solution to the above differential inequality with initial condition $e(t_i) = x(t_i) - x(t_i) = 0$ is

$$\|e(t)\| \leq \frac{\rho(\|x(t_i)\|) + \beta}{\|A\|}\left(\mathrm{e}^{\|A\|(t-t_i)} - 1\right).$$

Proof

Recall $x(t) = x(t_i) - e(t)$. Then, the above can be reduced to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\| =& \left\| Ax(t_i) - Ae(t) - B(c^\top B)^{-1}c^\top Ax(t_i) - B(c^\top B)^{-1}K\mathrm{sign}s(t_i) + Bd(t)\right\| \\ \le& \|A\|\|e(t)\| + \left\|\left(A - B(c^\top B)^{-1}c^\top A\right)x(t_i)\right\| + \|B(c^\top B)^{-1}K\| + \|B\|d_{\max} \\ =& \|A\|\,\|e(t)\| + \rho(\|x(t_i)\|) + \beta.\end{aligned}$$

The solution to the above differential inequality with initial condition $e(t_i) = x(t_i) - x(t_i) = 0$ is

$$\|e(t)\| \le \frac{\rho(\|x(t_i)\|) + \beta}{\|A\|}\left(\mathrm{e}^{\|A\|(t-t_i)} - 1\right).$$

At the triggering instant the error equals $\frac{\sigma\alpha}{\|c\|\|A\|}$. So,

$$\sigma\alpha \le \|c\|\,\|A\|\,\frac{\rho(\|x(t_i)\|) + \beta}{\|A\|}\left(\mathrm{e}^{\|A\|T_i} - 1\right).$$

On rearrangement, it gives the lower bound for T_i. It is seen that for any bounded region there exists a $\tau > 0$ such that $T_i > \tau$. Thus, the proof is completed.

SMC With Event-Triggering

Discussion

- α must be selected to incorporate practical constraints
- If $\tau_{\min}$ represents the minimum time for control execution, then α should correspond to $T_i > \tau_{\min}$
- The triggering rule

$$\|c\|\|A\|\|e\| < \sigma\alpha$$

demands continuous state measurement, thus it needs dedicated hardware

Self-Triggering Mechanism

This triggering mechanism is motivated by the event-triggering mechanism and development is based on it.

- Does not require hardware circuit to evaluate triggering condition
- Triggering instant is determined from previous sampled instant
- It is more feasible
- The self-triggering mechanism can be given as

$$t_{i+1} = t_i + \frac{1}{\|A\|} \ln\left(1 + \sigma \frac{\alpha}{\|c\|(\rho(\|x(t_i)\|) + \beta)}\right)$$

- This self-triggering scheme always guarantees positive lower bound for inter execution time

Example: Event-Triggering

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + 0.5\sin(10t)) .$$

We design $c = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$, $K = 0.65$, $\alpha = 0.5$, $\sigma = 0.85$.

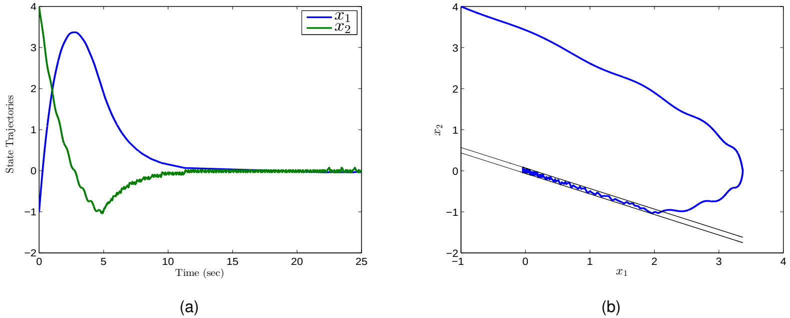
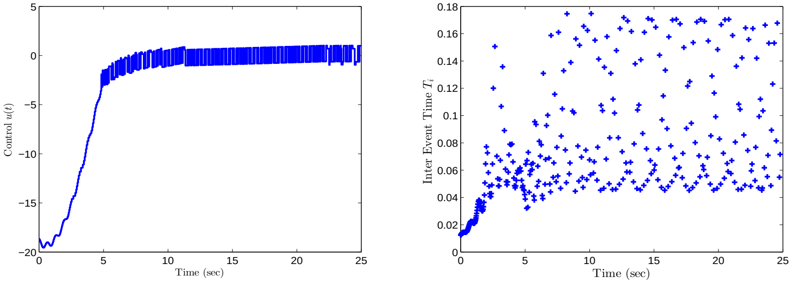


Figure: Performance of event-triggered SMC

Example: Event-Triggering

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + 0.5\sin(10t)) .$$

We design $c = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$, $K = 0.65$, $\alpha = 0.5$, $\sigma = 0.85$.



(a) (b)

Figure: Performance of event-triggered SMC

Outline

Definitions

- An event-triggered system with SMC is said to have robust global event-triggering property if

$$\inf_{\substack{x\in\mathbb{R}^n \\ i\in\mathbb{Z}_{\geq 0} \\ |d(t)|\leq d_0}} T_i > a_T.$$

- An event-triggered system with SMC is said to have robust semi-global event-triggering property if for any set $\mathcal{D}$ in $\mathbb{R}^n$,

$$\inf_{\substack{x\in\mathcal{D}\subset\mathbb{R}^n \\ i\in\mathbb{Z}_{\geq 0} \\ |d(t)|\leq d_0}} T_i > a_T.$$

- An event-triggered system with SMC is said to have robust local event-triggering property if for some subset $\mathcal{X}$ in $\mathcal{D} \subset \mathbb{R}^n$,

$$\inf_{\substack{x\in\mathcal{D}\cap\mathcal{X},\mathcal{X}\subset\mathcal{D}\subset\mathbb{R}^n \\ i\in\mathbb{Z}_{\geq 0} \\ |d(t)|\leq d_0}} T_i > a_T.$$

- From the above definition, we see that the triggering mechanism

$$t_{i+1} = \inf \{t > t_i \ : \ \|c\|\|A\|\|e(t)\| \geq \sigma\alpha\}$$

has robust semi-globally property. This can be concluded from

$$T_i \geq \frac{1}{\|A\|} \ln \left(1 + \sigma \frac{\alpha}{\|c\|(\rho(\|x(t_i)\|) + \beta)}\right).$$

- From the above definition, we see that the triggering mechanism

$$t_{i+1} = \inf \{t > t_i \ : \ \|c\|\|A\|\|e(t)\| \geq \sigma\alpha\}$$

has robust semi-globally property. This can be concluded from

$$T_i \geq \frac{1}{\|A\|} \ln \left(1 + \sigma \frac{\alpha}{\|c\|(\rho(\|x(t_i)\|) + \beta)}\right).$$

- The event-triggering mechanism can be designed to achieve globally robust stability of the event-triggered system.

Global Event-Triggered System

Global Triggering Rule

$$t_{i+1} = \inf \{t : t > t_i, \|c\| \|A\| \|e(t)\| \geq \sigma(\|x(t_i)\| + \alpha)\}$$

Global Triggering Rule

$$t_{i+1} = \inf \{ t : t > t_i, \|c\|\|A\|\|e(t)\| \geq \sigma(\|x(t_i)\| + \alpha) \}$$

Sliding Mode Control

The SMC for the global stability of event-triggered system is

$$u(t) = -(c^\top B)^{-1} \left(c^\top A x(t_i) + K(x(t_i)) \text{signs}(t_i) \right), \quad t \in [t_i, t_{i+1})$$

The switching gain $K(x(t_i))$ must satisfiy the following

1. $K(x(t_i)) > \left| c^\top B \right| d(t)$ for all $t \geq 0$,
2. $K(x(t_i)) > \|x(t_i)\| + \alpha$ for all $t \geq 0$.

We select it as

$$K(x(t_i)) = K_1 + K_2 \|x(t_i)\|$$

with $K_1 > \left| c^\top B \right| d_{\max} + \alpha$ and $K_2 > 1$.

Theorem

Let $\alpha > 0$ such that

$$\left|c^\top A e(t)\right| < \|x(t_i)\| + \alpha, \qquad t \geq 0.$$

Then, SMC guarantees that the sliding trajectory remains within a band

$$\left\{x \in \mathbb{R}^n : \left|c^\top x\right| < (\alpha + \|x(t_i)\|)\|A\|^{-1}\right\}$$

if

$$K(x(t_i)) > \sup_{t \geq 0} \left|c^\top B d(t)\right| + \|x(t_i)\| + \alpha.$$

Theorem

Let $\alpha > 0$ such that

$$\left|c^\top Ae(t)\right| < \|x(t_i)\| + \alpha, \qquad t \geq 0.$$

Then, SMC guarantees that the sliding trajectory remains within a band

$$\left\{x \in \mathbb{R}^n : \left|c^\top x\right| < (\alpha + \|x(t_i)\|)\|A\|^{-1}\right\}$$

if

$$K(x(t_i)) > \sup_{t\geq 0}\left|c^\top Bd(t)\right| + \|x(t_i)\| + \alpha.$$

Proof

Consider the Lyapunov function $V(s) = \frac{1}{2}s^2$. Differentiating V along the system trajectory

$$\begin{aligned}\dot{V}(s(t)) &= s\dot{s}\\ &= s(t)\big(c^\top Ax(t) - c^\top Ax(t_i) - K_1\text{sign}s(t_i)\\ &\quad - K_2\|x(t_i)\|\text{sign}s(t_i) + c^\top Bd\big).\end{aligned}$$

Proof

Recalling $e(t) = x(t_i) - x(t)$ and reducing the above further,

$$\begin{aligned}\dot{V}((t)) &= -s(t)c^\top Ae(t) - s(t)K_1\text{sign}s(t_i) - s(t)K_2\|x(t_i)\|\text{sign}s(t_i) + s(t)c^\top Bd(t)\\ &< |s(t)|\|x(t_i)\| + |s(t)|\alpha - s(t)K_1\text{sign}s(t_i) - s(t)K_2\|x(t_i)\|\text{sign}s(t_i)\\ &\quad + |s(t)|\left|c^\top B\right| d_{\max}.\end{aligned}$$

Until the sliding manifold is reached, we have $\text{sign}s(t_i) = \text{sign}s(t)$. So,

$$\begin{aligned}\dot{V}(s(t)) &< |s(t)|\|x(t_i)\| + |s(t)|\alpha - |s(t)|K_1 - |s(t)|K_2\|x(t_i)\| + |s(t)|\left|c^\top B\right| d_{\max}\\ &= -|s(t)|\left(K_1 - \alpha - \left|c^\top B\right| d_{\max}\right) - |s(t)|\left(K_2 - 1\right)\|x(t_i)\|\\ &< -|s(t)|\left(K_1 - \alpha - \left|c^\top B\right| d_{\max}\right)\\ &\leq -\eta|s(t)|\end{aligned}$$

for some $\eta > 0$. This shows the manifold is attractive.

Once the trajectory reaches the manifold, it may cross it as the control is not updated. But the trajectory remains bounded due to triggering mechanism.

Proof

Calculating the maximum deviation of sliding trajectory

$$\begin{aligned} |s(t_i) - s(t)| &= \left| c^\top x(t_i) - c^\top x(t) \right| \\ &\leq \|c\| \|e(t)\| \\ &< \frac{\alpha + \|x(t_i)\|}{\|A\|}. \end{aligned}$$

Thus, the sliding mode band can be obtained for the case $s(t_i) = 0$. Thus, the proof is completed.

Theorem

Let $\{t_i\}_{i=0}^{\infty}$ be the sequence of triggering instants generated by the global triggering rule. Then,

$$T_i \geq \frac{1}{\|A\|} \ln \left(1 + \sigma \frac{\|x(t_i)\| + \alpha}{\|c\|(\rho(\|x(t_i)\|) + \kappa)} \right)$$

where $\rho(\|x(t_i)\|)$ and κ are defined as

$$\rho(\|x(t_i)\|) := \left(\left\| A - B(c^\top B)^{-1} c^\top A \right\| + \left\| B(c^\top B)^{-1} K_2 \right\| \right) \|x(t_i)\|$$

and

$$\kappa := \left\| B(c^\top B)^{-1} K_1 \right\| + \|B\| d_{\max}.$$

Global Event-Triggered SMC

Theorem

Let $\{t_i\}_{i=0}^{\infty}$ be the sequence of triggering instants generated by the global triggering rule. Then,

$$T_i \geq \frac{1}{\|A\|} \ln \left(1 + \sigma \frac{\|x(t_i)\| + \alpha}{\|c\|(\rho(\|x(t_i)\|) + \kappa)} \right)$$

where $\rho(\|x(t_i)\|)$ and κ are defined as

$$\rho(\|x(t_i)\|) := \left(\left\| A - B(c^\top B)^{-1} c^\top A \right\| + \left\| B(c^\top B)^{-1} K_2 \right\| \right) \|x(t_i)\|$$

and

$$\kappa := \left\| B(c^\top B)^{-1} K_1 \right\| + \|B\| d_{\max}.$$

- See that $T_i > \tau$ for some $\tau > 0$ and all $x \in \mathbb{R}^n$

Proof

To show that T_i is strictly lower bounded from zero, we find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\| &\le \left\|\frac{\mathrm{d}e(t)}{\mathrm{d}t}\right\| = \left\|\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\| \\
&= \left\| Ax(t) - B(c^\top B)^{-1}c^\top Ax(t_i) - B(c^\top B)^{-1}\left(K_1 + K_2\|x(t_i)\|\right)\mathrm{sign}s(t_i) + Bd(t)\right\| \\
&\le \Big\| Ax(t_i) - Ae(t) - B(c^\top B)^{-1}c^\top Ax(t_i) - B(c^\top B)^{-1}K_1\mathrm{sign}s(t_i) \\
&\quad - B(c^\top B)^{-1}K_2\|x(t_i)\|\mathrm{sign}s(t_i) + Bd(t)\Big\| \\
&\le \|A\|\|e(t)\| + \|A - B(c^\top B)^{-1}c^\top A\|\|x(t_i)\| + \|B(c^\top B)^{-1}K_2\|\|x(t_i)\| \\
&\quad + \|B(c^\top B)^{-1}K_1\| + \|B\|d_{\max} \\
&= \|A\|\|e(t)\| + \rho(\|x(t_i)\|) + \kappa.
\end{aligned}
$$

Proof

To show that T_i is strictly lower bounded from zero, we find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\| &\leq \left\|\frac{\mathrm{d}e(t)}{\mathrm{d}t}\right\| = \left\|\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\| \\
&= \left\|Ax(t) - B(c^\top B)^{-1}c^\top Ax(t_i) - B(c^\top B)^{-1}\left(K_1 + K_2\|x(t_i)\|\right)\mathrm{sign}s(t_i) + Bd(t)\right\| \\
&\leq \Big\|Ax(t_i) - Ae(t) - B(c^\top B)^{-1}c^\top Ax(t_i) - B(c^\top B)^{-1}K_1\mathrm{sign}s(t_i) \\
&\quad - B(c^\top B)^{-1}K_2\|x(t_i)\|\mathrm{sign}s(t_i) + Bd(t)\Big\| \\
&\leq \|A\|\|e(t)\| + \|A - B(c^\top B)^{-1}c^\top A\|\|x(t_i)\| + \|B(c^\top B)^{-1}K_2\|\|x(t_i)\| \\
&\quad + \|B(c^\top B)^{-1}K_1\| + \|B\|d_{\max} \\
&= \|A\|\|e(t)\| + \rho(\|x(t_i)\|) + \kappa.
\end{aligned}
$$

Solving the above differential inequality with $\|e(t_i)\| = 0$,

$$
\|e(t)\| \leq \frac{\rho(\|x(t_i)\|) + \kappa}{\|A\|}\left(\mathrm{e}^{\|A\|(t-t_i)} - 1\right)
$$

for $t \in [t_i, t_{i+1})$.

At the triggering instant,

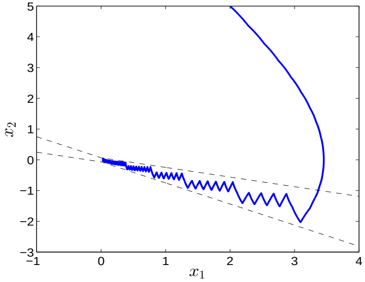
$$\frac{\sigma(\|x(t_i)\| + \alpha)}{\|c\|\|A\|} \leq \frac{\rho(\|x(t_i)\|) + \kappa}{\|A\|}\left(\mathrm{e}^{\|A\|T_i} - 1\right).$$

On rearrangement, it gives the lower bound for inter event-time. Thus, the proof is completed.

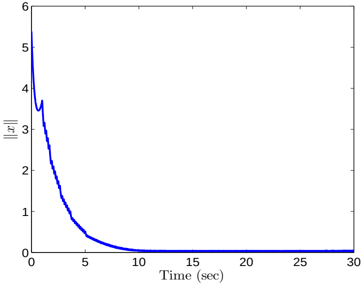
Example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + 0.5\sin(10t)) \,.$$

We design $c = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$, $K_1 = 0.7$, $K_2 = 1.2$, $\alpha = 0.5$, $\sigma = 0.85$.



(a)



(b)

Figure: Performance of event-triggered SMC

Example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + 0.5\sin(10t))\,.$$

We design $c = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$, $K_1 = 0.7$, $K_2 = 1.2$, $\alpha = 0.5$, $\sigma = 0.85$.

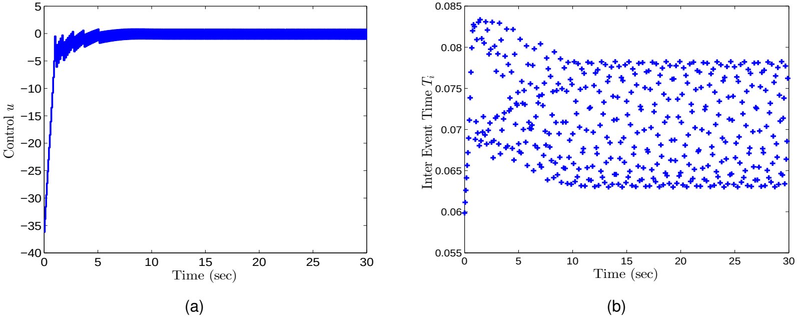


Figure: Performance of event-triggered SMC

Outline

Conclusion

- In event-triggered SMC, the steady-state bound can be designed as per any desired value
- Robust system performance is also achieved
- This strategy can be extended to nonlinear systems
- Global stability of the event-triggered system is also obtained by designing the event-triggering rule

Thank you for Attention

Questions?