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Terminal Sliding Mode Control

Classical sliding Mode Control

- Most commonly used switching manifolds are the linear hyperplanes which guarantee asymptotic stability of the system motion during sliding mode
- System states reach the equilibrium in infinite time

Motivation

- Need to improve the system performance during sliding mode
- System should reach equilibrium point as fast as possible
- A non-Lipschitz nonlinear manifold has faster convergence near the equilibrium point (i.e., origin)

Terminal Sliding Mode

- In TSM, a nonlinear sliding surface is proposed
- The equilibrium is a terminal attractor, i.e., the states can be reached in finite time and are stable
- The term terminal is referred to the equilibrium which can be reached in finite time and is stable

Terminal Sliding Mode

Consider a second order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u$$

where $g^{-1}(x) \neq 0$. Select the TSM manifold as

$$s = x_2 + \beta x_1^{q/p}, \quad \beta > 0$$

where p and q are odd integers such that $q < p$. Differentiating s, we obtain

$$\dot{s} = \dot{x}_2 + \beta \frac{q}{p} x_1^{\frac{q}{p}-1} \dot{x}_1$$
$$= f(x) + g(x)u + \beta \frac{q}{p} x_1^{\frac{q-p}{p}} x_2.$$

Design the control as $u = -g^{-1}(x)\left(f(x) + \beta \frac{q}{p} x_1^{\frac{q-p}{p}} x_2 + K sign(s)\right)$ where $K > 0$.

Then, simplifying further

$$\dot{s} = -K sign(s).$$

Terminal Sliding Mode: Concept

Reduced Order Dynamics

During the sliding mode, we achieve $s = 0$, that implies

$$\dot{x}_1 = x_2 = -\beta x_1^{q/p}.$$

Now solving for time t_1 such that $x_1(t) = 0$ for all $t \geq t_1$ given any initial condition $x_1(t_0)$ at time $t = t_0$

$$\begin{aligned} t_1 &= t_0 - \frac{1}{\beta} \int_{x_1(t_0)}^{0} x_1^{-\frac{q}{p}} \mathrm{d}x_1 \\ &= t_0 + \frac{p}{\beta(p-q)} x_1^{\frac{p-q}{p}}(t_0). \end{aligned}$$

Note that $p - q$ is an even number. It implies that x_1 goes to zero in time t_1 and remains there for all time $t \geq t_1$ since $\dot{x}_1 = 0$. Then, $x_2 = 0$ and thus, finite time stability is ensured.

Remarks

- The control expression contains negative exponent of x_1, so it becomes unbounded for $x_1 = 0$
- However, during sliding the control can be bounded by selecting $\frac{q}{p} \in [0.5, 1)$. It can

Terminal Sliding Manifold

Consider the terminal sliding manifold

$$s = x_2 + \beta x_1^{q/p}.$$

Remarks

- TSM manifold is a non-Lipschitz in nature
- Near origin the convergence rate is much faster than the linear surface
- Solution of the such system reach the equilibrium point in finite time
- Solution in forward time direction is unique
- If $p = q$, then TSM manifold becomes linear sliding surface

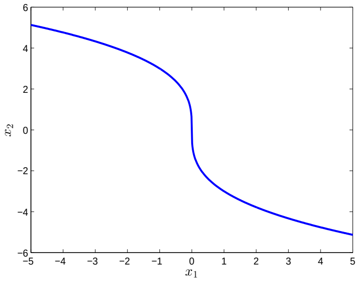


Figure: Terminal sliding manifold

TSM for SISO System

SISO System

$$\dot{x}_i = x_{i+1} \quad i = 1, 2, \ldots, n-1,$$
$$\dot{x}_n = f(x) + g(x)u.$$

For this n^{th} order SISO system, hierarchical TSM manifolds are defined as

$$\begin{aligned}
s_1 &= \dot{s}_0 + \beta_1 s_0^{q_1/p_1} \\
s_2 &= \dot{s}_1 + \beta_2 s_1^{q_2/p_2} \\
&\vdots \\
s_{n-2} &= \dot{s}_{n-3} + \beta_{n-2} s_{n-3}^{q_{n-2}/p_{n-2}} \\
s_{n-1} &= \dot{s}_{n-2} + \beta_{n-1} s_{n-2}^{q_{n-1}/p_{n-1}}
\end{aligned}$$

where $s_0 = x_1$, $\beta_i > 0$, $p_i > q_i$ and p_i, q_i are positive odd integers. The values of integer must satisfy for bounded control during sliding given as[a]

$$\frac{q_k}{p_k} > \frac{n-k}{n-k+1} \quad k = n-1, \ldots, 1.$$

TSM Control

Differentiating s_{n-1}, we obtain

$$\dot{s}_{n-1} = \ddot{s}_{n-2} + \beta_{n-1}\frac{q_{n-1}}{p_{n-1}} s_{n-2}^{\frac{q_{n-1}}{p_{n-1}}-1}\frac{\mathrm{d}}{\mathrm{d}t}s_{n-2}$$
$$= f(x) + g(x)u + \sum_{i=1}^{n-1}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\beta_{n-i}s_{n-i-1}^{\frac{q_{n-i}}{p_{n-i}}}.$$

Now, design the control law

$$u = -g^{-1}(x)\left(f(x) + \sum_{i=1}^{n-1}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\beta_{n-i}s_{n-i-1}^{\eta_{n-i}} + K\mathrm{sign}(s_{n-1})\right).$$

Substituting u in the above dynamics

$$\dot{s}_{n-1} = -K\mathrm{sign}(s_{n-1}).$$

It leads to $s_{n-1} = 0$ in finite time. This implies $s_{n-2} = 0$ and subsequently to $s_0 = x_1 = 0$. Thus, all the states of the goes to zero in finite time.

Fast Terminal Sliding Mode Control

Fast Terminal Sliding Mode

Motivation

Consider the terminal sliding manifold

$$s = x_2 + \beta x_1^{q/p}.$$

During sliding $\dot{x}_1 = -\beta x_1^{\frac{q}{p}}$. It can be observed that

- if the initial condition is far away from the origin the term $x_1^{\frac{q}{p}}$ has lesser magnitude than that of linear counter part
- convergence can be enhanced by incorporating a linear term in terminal sliding manifold

Fast TSM

To achieve faster convergence, a new TSM manifold is defined as

$$s = x_2 + \alpha x_1 + \beta x_1^{q/p}$$

and during sliding $\dot{x}_1 = -\alpha x_1 - \beta x_1^{\frac{q}{p}}$. Thus,

- when x_1 is far away from the origin αx_1 dominates, in other words, $\dot{x}_1 \approx -\alpha x_1$, so convergence is faster.

Reduced Order System

The reduced system during sliding can be given as

$$\dot{x}_1 = -\alpha x_1 - \beta x_1^{\frac{q}{p}}.$$

The time of convergence of x_1 to zero can be obtained as

$$\begin{aligned} t_1 &= t_0 + \int_{x_1(t_0)}^{0} \frac{\mathrm{d}x_1}{-\alpha x_1 - \beta x_1^{\frac{q}{p}}} \\ &= t_0 + \int_{x_1(t_0)}^{0} \frac{\mathrm{d}x_1}{-x_1^{\frac{q}{p}} \left(\alpha x_1^{1-\frac{q}{p}} + \beta\right)} \\ &= t_0 + \frac{p}{\alpha(p-q)} \left(\ln \left(\alpha x_1^{\frac{p-q}{p}}(t_0) + \beta \right) - \ln(\beta) \right) \end{aligned}$$

where t_0 is the time taken by the system to reach the fast terminal sliding manifold.

SISO System

$$\dot{x}_i = x_{i+1} \quad i = 1, 2, \ldots, n-1,$$
$$\dot{x}_n = f(x) + g(x)u.$$

For this n^{th} order SISO system, hierarchical TSM manifolds are defined as

$$s_1 = \dot{s}_0 + \alpha_1 s_0 + \beta_1 s_0^{q_1/p_1}$$
$$s_2 = \dot{s}_1 + \alpha_2 s_1 + \beta_2 s_1^{q_2/p_2}$$
$$\vdots$$
$$s_{n-2} = \dot{s}_{n-3} + \alpha_{n-2} s_{n-3} + \beta_{n-2} s_{n-3}^{q_{n-2}/p_{n-2}}$$
$$s_{n-1} = \dot{s}_{n-2} + \alpha_{n-1} s_{n-2} + \beta_{n-1} s_{n-2}^{q_{n-1}/p_{n-1}}$$

where $s_0 = x_1$, $\beta_i > 0$, $p_i > q_i$ and p_i, q_i are positive odd integers. The values of integer must satisfy for bounded control during sliding given as[a]

$$\frac{q_k}{p_k} > \frac{n-k-1}{n-k} \quad k = n-1, \ldots, 1.$$

Non Singular Terminal Sliding Mode Control

Non Singular Terminal Sliding Mode

Terminal Sliding Mode Control

Recall the TSM control law

$$u = -g^{-1}(x)\left(f(x) + \beta \frac{q}{p} x_1^{\frac{q}{p}-1} x_2 + K sign(s)\right).$$

We see that the exponent of x_1 is $\frac{q}{p} - 1 < 0$. So, when system trajectories crosses $x_1 = 0$ axis, then control law become infinite. Such a controller can not be applied to the system and it is called singularity in the TSM.

Non Singular Terminal Sliding Mode

To avoid such a situation, a new terminal manifold is proposed called non singular terminal sliding mode (NTSM)[a]

$$s = x_1 + \frac{1}{\beta^{\frac{p}{q}}} x_2^{\frac{p}{q}}, \quad 1 < \frac{p}{q} < 2$$

The TSM and NTSM surfaces are equivalent to each other when $s = 0$.

[a] Y. Feng, X. Yu and Z. Man, "Non-singular terminal sliding mode control of rigid manipulators", Automatica, vol. 38, no. 12, pp. 2159–2167, 2002.

Equivalence Between TSM and NTSM

- It is to be noted that $x_1^{\frac{q}{p}}$ is an odd function, i.e., $(-x_1)^{\frac{q}{p}} = -x_1^{\frac{q}{p}}$.
- One way to realize this, we can take $x_1^{\frac{q}{p}} = |x_1|^{\frac{q}{p}}\mathrm{sign}(x_1)$.

Now, we shall see equivalence between TSM and NTSM when $s = 0$. From NTSM with $s = 0$, we have

$$x_1 = -\frac{1}{\beta^{\frac{p}{q}}}|x_2|^{\frac{p}{q}}\mathrm{sign}(x_2).$$

From this, we conclude that $\mathrm{sign}(x_1) = -\mathrm{sign}(x_2)$. Multiplying both sides by $\mathrm{sign}(x_1)$ and then taking $\frac{q}{p}$ power on both sides (use the fact $|x_1| = x_1\mathrm{sign}(x_1)$)

$$\beta|x_1|^{\frac{q}{p}} = |x_2|.$$

Multiplying both sides by $\mathrm{sign}(x_2)$, it yields

$$-\beta|x_1|^{\frac{q}{p}}\mathrm{sign}(x_1) = x_2.$$

This in other words equal to $x_2 = -\beta x_1^{\frac{q}{p}}$. Thus, time taken by the system to reach $x_1 = 0$ is same as that of TSM.

Finite Time Reachability to NTSM Manifold

Differentiating s

$$\begin{aligned}\dot{s} &= \dot{x}_1 + \frac{1}{\beta^{\frac{p}{q}}}\frac{p}{q}x_2^{\frac{p}{q}-1}\dot{x}_2 \\ &= x_2 + \frac{1}{\beta^{\frac{p}{q}}}\frac{p}{q}x_2^{\frac{p}{q}-1}(f(x)+g(x)u).\end{aligned}$$

Design the control law as given below

$$u = -g^{-1}(x)\left(f(x) + \beta^{\frac{p}{q}}\frac{q}{p}x_2^{2-\frac{p}{q}} + K\text{sign}(s)\right).$$

Substituting for u in the $\dot{s}$, we obtain

$$\dot{s} = -\frac{1}{\beta^{\frac{p}{q}}}\frac{p}{q}x_2^{\frac{p}{q}-1}K\text{sign}(s).$$

To show convergence to origin, we consider $V = \frac{1}{2}s^2$. Differentiating V along the system trajectories

$$\dot{V} = s\dot{s} = -s\left(\frac{1}{\beta^{\frac{p}{q}}}\frac{p}{q}x_2^{\frac{p}{q}-1}K\text{sign}(s)\right)$$

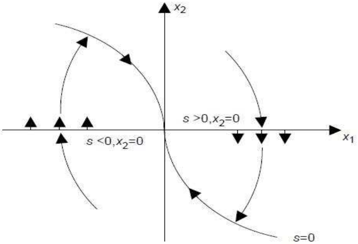
Finite Time Reachability to NTSM Manifold

which on further simplification

$$\dot{V} = -\frac{1}{\beta^{\frac{p}{q}}}\frac{p}{q}x_2^{\frac{p}{q}-1}K|s|.$$

Define $\rho(x_2) := \frac{1}{\beta^{\frac{p}{q}}}\frac{p}{q}x_2^{\frac{p}{q}-1}K$. Then $s\dot{s} = -\rho(x_2)|s|$. If $x_2 \neq 0$, we have $\rho(x_2) > 0$. That means, the trajectories are attracted towards the NTSM manifold and hence finite time convergence is achieved. For $x_2 = 0$, we write

$$\dot{x}_2 = -K\mathrm{sign}(s).$$



Finite Time Reachability to NTSM Manifold

- If $s > 0$, then $\dot{x}_2 = -K$. Similarly for $s < 0$, we have $\dot{x}_2 = K$.
- It implies that there exists a small vicinity $|x_2| < \delta$ around $x_2 = 0$ such that for $s > 0$, we have $\dot{x}_2 = -K$. Similarly for $s < 0$.
- Then x_2 decreases for $s > 0$ and increases for $s < 0$. So, the sliding trajectories will cross the boundaries $x_2 = \delta$ and $x_2 = -\delta$ in finite time and similarly for $s < 0$.
- Therefore, the trajectories are attracted towards the NTSM manifold in finite time. Thus, proof is completed.

Prescribed Convergence Law

Second Order System

Consider a second order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u$$

where $g(x) \neq 0$. Define the sliding variable as $s = x_2 + \beta |x_1|^{\frac{1}{2}} \mathrm{sign}(x_1)$. The control law is given as

$$u = -g^{-1}(x)(f(x) + \alpha \mathrm{sign}(s))$$

where $\alpha > \frac{\beta^2}{2}$. Substituting the control in the system dynamics, we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha \mathrm{sign}(s).$$

It can be seen that trajectories are driven by a constant rate gain, hence the name prescribed convergence law[a][b].

[a] A. Levant, "Universal single-input-single-output(SISO) sliding mode controllers with finite time convergence", IEEE Trans. Autom. Control, vol. 46, no. 9, pp. 1447–1451, 2001.

[b] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control", Int. J. Control, vol. 76, no. 9/10, pp. 924–941, 2003.

Proof of Prescribed Convergence Law

Differentiating s and substituting for u

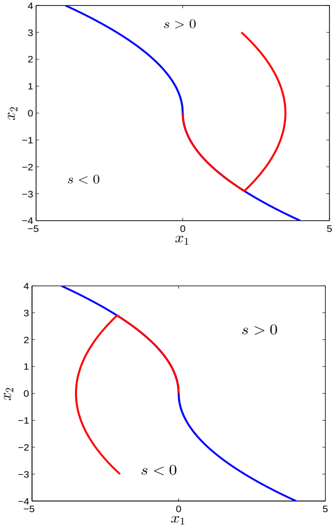
$$\dot{s} = -\alpha \mathrm{sign}(s) + \frac{1}{2}\beta|x_1|^{-\frac{1}{2}}x_2.$$

It can be noted that the initial conditions may be located either in $s > 0$ or $s < 0$ ($s = 0$ is trivial). Consider $s > 0$ and then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha.$$

Due to geometric reason, the system trajectories decreases and eventually hit the curve $s = 0$ on the way. Similarly, for the case $s < 0$ as the dynamics takes the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \alpha.$$



Prescribed Convergence Law

Proof of Prescribed Convergence Law

When $s = 0$, we have $x_2 = -\beta|x_1|^{\frac{1}{2}}\mathrm{sign}(x_1)$, so

$$\begin{aligned}\dot{s} &= -\alpha\mathrm{sign}(s) - \frac{1}{2}\beta^2\mathrm{sign}(x_1)\\ &\leq -\eta\mathrm{sign}(s).\end{aligned}$$

Thus, once the trajectories hit $s = 0$, it can never leave it provided $\alpha > \frac{\beta^2}{2}$ and hence, the sliding mode is enforced in finite time.

System Dynamics

During sliding, we obtain $\dot{x}_1 = -\beta|x_1|^{\frac{1}{2}}\mathrm{sign}(x_1)$. Consider $V = \frac{1}{2}x_1^2$. Then,

$$\begin{aligned}\dot{V} = x_1\dot{x}_1 &= -x_1\beta|x_1|^{\frac{1}{2}}\mathrm{sign}(x_1) = -\beta|x_1|^{\frac{3}{2}}\\ &= -\beta 2^{\frac{3}{4}}V^{\frac{3}{4}}.\end{aligned}$$

We see that V goes to zero in time $t_1 = t_0 + \frac{4}{\beta 2^{3/4}}V^{\frac{1}{4}}(t_0)$. Thus, finite time stability is ensured.

Summary

Remarks

- Prescribed convergence law and TSM are similar except in their control structures.
- NTSM is proposed to avoid the singularity issue in TSM.
- There is no singularity in the prescribed convergence law.
- These all control structures belong to second-order sliding mode control.

Discrete Terminal Sliding Mode Control

Terminal Sliding Mode: Recap

- Sliding mode control concept which tries to make $x = 0$ in finite time (not just $s = 0$).
- In continuous time it is accomplished by using a non linear sliding surface of form (given here for 2nd order).

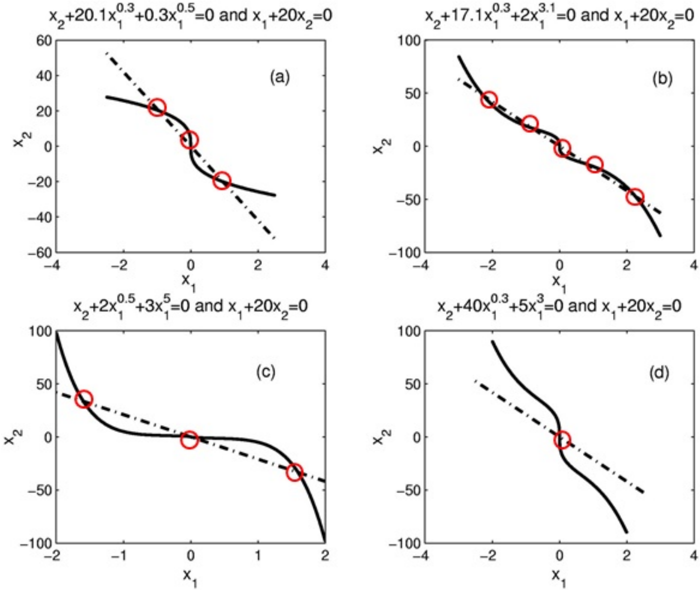
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
s &= x_2 + \alpha x_1^{\gamma} + \beta x_1^{\rho}, \\
&\alpha, \beta, \gamma, \rho > 0, \\
&0 < \gamma < 1 \\
&\gamma < \rho. \\
&\gamma, \rho \rightarrow p/q; \quad p, q \quad odd
\end{aligned}$$

Discretization

- Let us assume the system is discretized at a sampling interval τ, and the system state is moving along the sliding surface (somehow).
- If there is a k^* such that $x = 0$ after k^*, then

$$
\begin{aligned}
0 &= x_1(k^* + 1) = x_1(k^*) + \tau x_2(k^*) \\
0 &= s(k^*) = x_2(k^*) + \alpha x_1^{\gamma}(k^*) + \beta x_1^{\rho}(k^*)
\end{aligned}
$$

The possibilities



The Improbability

- There are only a finite number of points from which the states can go to origin.
- It is highly unlikely that the system would cross these points.
- Hence, it can be assumed that due to discretization, the finite-time part of terminal sliding mode is no longer true.

- Analysing around the origin, the discrete-time system

$$f(x_1(k)) = x_1(k+1) = x_1(k) - \alpha x_1^{\gamma}(k) + \beta x_1^{\rho}(k)$$

it is found that

$$\left|\frac{df(y)}{dy}\right|_{y\to 0} = |1 - \alpha\tau\gamma y^{\gamma-1} - \beta\tau\rho y^{\rho-1}|_{y\to 0} = \infty > 1$$

- It is required that $\left|\frac{df(y)}{dy}\right| < 1$, so system is unstable around origin and it diverges from origin.

Periodicity

- Analysis shows that if $f(y^*) = -y^*$, then $\{y^*, -y^*\}$ form a limit set. (Not much can be said in this case).
- Further, this is the only possible 2 period.

2-period analysis

- Let us assume that x^* is a point that has a 2-period motion. Then, $f(f(x^*)) = x^*$, *i.e*

$$x^* - \alpha\tau(x^*)^\gamma - \beta\tau(x^*)^\rho - \alpha\tau(f(x^*))^\gamma - \beta\tau(f(x^*))^\rho$$

- Or equivalently

$$(\alpha\tau(f(x^*))^\gamma + \beta\tau(f(x^*))^\rho) = -(\alpha\tau(x^*)^\gamma + \beta\tau(x^*)^\rho) \quad = \alpha\tau(-x^*)^\gamma + \beta\tau(-x^*)^\rho$$

- Using the fact that $(\alpha\tau(x^*)^\gamma + \beta\tau(x^*)^\rho)$ is monotonic it can be said that $f(y^*) = -y^*$ is the only 2-periodic orbit possible (if it exists).

Condition for Stability of 2-period orbits

- Consider there exists a 2-period orbit satisfying $f(x^*) = -x^*$,
- Using the discrete system stability condition around $x = x*$, the stability of the 2-period can be assured if

$$\left|\frac{df(f(x))}{dx}\right|_{x=x^*} = \left|\frac{df(x)}{dx}\right|_{x=f(x^*)}\left|\frac{df(x)}{dx}\right|_{x=x^*} < 1$$
$$= (1 - \alpha\tau\gamma(x^*)^{\gamma-1} - \beta\tau\rho(x^*)^{\rho-1})^2 < 1$$
$$-2 < -\alpha\tau\gamma(x^*)^{\gamma-1} - \beta\tau\rho(x^*)^{\rho-1} < 0$$

Conditions

- From the conditions imposed on γ, ρ it can be said that

$$-\alpha\tau\gamma(x^*)^{\gamma-1} - \beta\tau\rho(x^*)^{\rho-1} < 0$$

- Thus, stability condition reduces to

$$-2 < -\alpha\tau\gamma(x^*)^{\gamma-1} - \beta\tau\rho(x^*)^{\rho-1} \quad (1)$$

- Condition (1) further reduces to

$$2 > \alpha\tau\gamma(x^*)^{\gamma-1} + \beta\tau\rho(x^*)^{\rho-1} \quad (2)$$

Conditions

- As said earlier, 2-period orbits are only those satisfying

$$f(x^*) = -x^*$$

- Thus,

$$x^* - \alpha\tau(x^*)^\gamma - \beta\tau(x^*)^\rho = -x^*$$

- which cab be simplified into

$$\alpha\tau(x^*)^{\gamma-1} + \beta\tau(x^*)^{\rho-1} = 2$$

  for $x^* \neq 0$.

- Substituting the LHS of above equation instead of 2 in the inequality (2), we get

$$\alpha\tau(x^*)^{\gamma-1} + \beta\tau(x^*)^{\rho-1} > \alpha\tau\gamma(x^*)^{\gamma-1} + \beta\tau\rho(x^*)^{\rho-1} \tag{3}$$

Conditions

- Now due to the restrictions on γ and ρ, as defined earlier, we can write

$$(x^*)^{\gamma-1} = |x^*|^{\gamma-1}, \;\; (x^*)^{\rho-1} = |x^*|^{\rho-1}$$

  which avoids complex case of x^*.
- Now dividing (3) by $\tau|x^*|^{\gamma-1}$ and rearranging, we get

$$\alpha(1-\gamma) > \beta(\rho-1)|x^*|^{\rho-\gamma} \tag{4}$$

- Hence for $\rho > 1$ the condition for stable 2-period can thus be derived to be

$$\frac{\alpha(1-\gamma)}{\beta(\rho-1)} > |x^*|^{\rho-\gamma}, \qquad \rho > 1 \tag{5}$$

- If $\rho \leq 1$, we get $\beta(\rho-1)|x^*|^{\rho-\gamma} < 0$ and $\alpha(1-\gamma) > 0$ for all x^*.
- Thus, in case of $\rho \leq 1$, there is no extra condition other than $f(x^*) = -x^*$ for existence of stable 2-period orbits.

Otherwise

- If no such y^* exists, then there are no periodic orbits (Sarkovskii Theorem).
- Sarkovskii Theorem :
  - The existence of a period i orbit implies the existence of all periodic orbits of period j where j follows i in the table.
  - The non existence of a period j orbit would imply the non existence of a period i orbit where i precedes j in the table.

$$\begin{array}{ccccc} 3 & 5 & 7 & 9 & \cdots \\ 6 & 10 & 14 & 18 & \cdots \\ \vdots & & & & \\ 2^n 3 & 2^n 5 & 2^n 7 & 2^n 9 & \cdots \\ 2^n & 2^{n-1} & \cdots & 2 & 1 \end{array}$$

- Consider there is no 2-period orbit (stable or unstable) existing in the system.
- Since system is not stable around origin (the only stationary point), the system would diverge. (while still on the sliding surface).

Analysis

- Discretization of continuous terminal sliding mode.
  - Almost never leads to finite time convergence.
  - Certainly leads to an instability around origin.
  - May lead to periodic / chaotic behavior (Chaotic behavior can exist only if periodic behavior is also possible).
  - Failing which system is unstable
- Discrete-time terminal sliding mode should be handled differently from continuous-time terminal sliding mode.

Aim

Given a discrete-time system,

$$x(k+1) = F(x(k, u(k))$$

the terminal sliding surface is such that the the system dynamics confined to the surface (brought about by control) has the property

$$x(k+1) = F_c(x(k))$$
$$x(k+k_d) = F_c^{k_d}(x(k)) = 0, \quad k_d < \infty \Rightarrow \textit{nilpotent} \quad \textit{function}$$

Algorithm for Discrete TSM

- Using appropriate transformation ψ, transform the system into Brunowsky canonical form,

$$z_i(k+1) = z_{i+1}(k), \quad i = 1, 2, \cdots, n$$
$$z_n(k+1) = a_d x(k) + b_d u(k)$$

- Sliding surface is $z_n(k) = 0$.
- Reaching law is $z_n(k+1) = 0$.
- Design appropriate control to achieve DSM.
- It is to be noted that control should not be based on continuous SMC idea (Bartoszewicz, Bartolini-Utkin).
- Can be converted to MROF also.

Example

- Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} x_2 + f_x^2(k) \\ f_x(k) \\ x_1(k) + 2x_2(k)f_x^2(k) + f_x^4(k) \end{bmatrix}, \qquad f_x(k) = x_3(k) - x_1^2(k) + u(k)$$

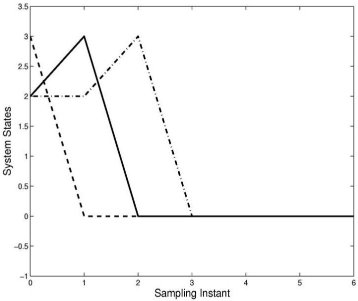
- In a transformed co-ordinate frame with

$$z(k) = \begin{bmatrix} x_3(k) - x_1^2(k) \\ x_1(k) - x_2^2(k) \\ x_2(k) \end{bmatrix}$$

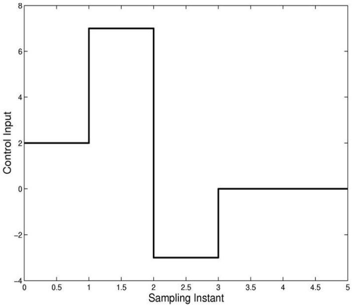
we have

$$z(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Results



System States
Sampling Instant



Control Input
Sampling Instant

Terminal Sliding Mode

- In TSM, a nonlinear sliding surface is proposed
- The equilibrium is a terminal attractor, i.e., the states can be reached in finite time and are stable
- The term terminal is referred to the equilibrium which can be reached in finite time and is stable

Discrete Terminal Sliding Mode

- Finite-time convergence of system states are not ensured
- Results in periodic motion
- Established only period-2 motion in steady-state

Terminal Sliding Mode: Concept

Terminal Sliding Mode

Consider a second order system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= f(x) + g(x)u\end{aligned}$$

where $g^{-1}(x) \neq 0$. Select the TSM manifold as

$$s = x_2 + \beta x_1^{q/p}, \quad \beta > 0$$

where p and q are odd integers such that $q < p$. Differentiating s, we obtain

$$\begin{aligned}\dot{s} &= \dot{x}_2 + \beta \frac{q}{p} x_1^{\frac{q}{p}-1} \dot{x}_1\\ &= f(x) + g(x)u + \beta \frac{q}{p} x_1^{\frac{q-p}{p}} x_2.\end{aligned}$$

The control as $u = -g^{-1}(x)\left(f(x) + \beta \frac{q}{p} x_1^{\frac{q-p}{p}} x_2 + K sign(s)\right)$ results finite-time stability of

$$\dot{s} = -K sign(s), \qquad K > 0.$$

Discrete TSM

Discretized Plant

Consider Euler discretization of the continuous-time system

$$x_1(k+1) = x_1(k) + hx_2(k)$$
$$x_2(k+1) = x_2(k) + hf(x(k)) + hg(x(k))u(k)$$

and the sliding manifold as $s(k) = x_2(k) + \beta x_1^{\eta}(k)$. If the control is chosen such that $s(k+1) = 0$ for all k, then

$$\Phi(x_1) = x_1(k+1) = x_1(k) - h\beta x_1^{\eta}(k).$$

- The stability of the system is given by the solution of $\Phi(x_1)$
- It has been shown that it results periodic solutions
- To guarantee the stability of the system, all the possible periodic orbits are found

Periodic Orbits

Period-1 Orbit

There exits only period-1 is $\Phi(x_1(k)) = x_1(k)$ if $x_1(k) = 0$ and it is seen that this point is unstable. To see this

$$\Phi(x_1(k)) = x_1(k) - h\beta x_1^{\eta}(k) = x_1(k) \quad \implies x_1(k) = 0.$$

Period-2 Orbit

For period-2 point there exists a point $x_2^{(1)}$ such that $\Phi^2(x_2^{(1)}) = x_2^{(1)}$ i.e., $\Phi(x_2^{(1)}) = x_2^{(2)}, \Phi(x_2^{(2)}) = x_2^{(1)}$, then

$$x_2^{(2)} = x_2^{(1)} - h\beta \left\{ (x_2^{(1)})^{\eta} \right\}$$
$$x_2^{(1)} = x_2^{(1)} - h\beta \left\{ (x_2^{(1)})^{\eta} + (x_2^{(2)})^{\eta} \right\}$$

and further

$$(x_2^{(2)})^{\eta} = -(x_2^{(1)})^{\eta}.$$

Since $(x_2^{(1)})^{\eta}$ is an odd function, then $x_2^{(2)} = -x_2^{(1)}$ is the solution. Then period-2 points can be given as $\{(x_2^{(1)}, -2x_2^{(1)}/h), (-x_2^{(1)}, 2x_2^{(1)}/h)\}$ and the limit set as $\{x_2^{(1)}, -x_2^{(1)}\}$.

Period-4 Orbit

Let $x_4^{(1)}$, $x_4^{(2)}$, $x_4^{(3)}$ and $x_4^{(4)}$ be the four points such that it satisfies period-4 motion, i.e., $\Phi(x_4^{(1)}) = x_4^{(2)}$, $\Phi(x_4^{(2)}) = x_4^{(3)}$, $\Phi(x_4^{(3)}) = x_4^{(4)}$, $\Phi(x_4^{(4)}) = x_4^{(1)}$, then

$$
\begin{aligned}
x_4^{(2)} &= x_4^{(1)} - h\beta \left\{ (x_4^{(1)})^\eta \right\} \\
x_4^{(3)} &= x_4^{(1)} - h\beta \left\{ (x_4^{(1)})^\eta + (x_4^{(2)})^\eta \right\} \\
x_4^{(4)} &= x_4^{(1)} - h\beta \left\{ (x_4^{(1)})^\eta + (x_4^{(2)})^\eta + (x_4^{(3)})^\eta \right\} \\
x_4^{(1)} &= x_4^{(1)} - h\beta \left\{ (x_4^{(1)})^\eta + (x_4^{(2)})^\eta + (x_4^{(3)})^\eta + (x_4^{(4)})^\eta \right\}.
\end{aligned}
$$

Thus, we obtain the relation

$$(x_4^{(1)})^\eta + (x_4^{(2)})^\eta + (x_4^{(3)})^\eta + (x_4^{(4)})^\eta = 0$$

and

$$
\begin{aligned}
(x_4^{(1)})^\eta + (x_4^{(2)})^\eta &= -(x_4^{(3)})^\eta - (x_4^{(4)})^\eta \\
&= (-x_4^{(3)})^\eta + (-x_4^{(4)})^\eta.
\end{aligned}
$$

Periodic Points

Period-4 Orbit

Due to odd nature of the function $\Phi(x(k))$, we arrive at

$$x_4^{(1)} = -x_4^{(3)} \quad \text{and} \quad x_4^{(2)} = -x_4^{(4)}.$$

The period-4 motion can be given as $\mathcal{O}^4 = \{(x_4^{(1)}, (x_4^{(2)} - x_4^{(1)})/h), (x_4^{(2)}, -(x_4^{(2)} + x_4^{(1)})/h), (-x_4^{(1)}, -(x_4^{(2)} - x_4^{(1)})/h), (-x_4^{(2)}, (x_4^{(2)} + x_4^{(1)})/h)\}$.

Period-2m Orbit

The period-2m would have in general the periodic motion restricted on the set given as $\mathcal{O}^{2m} = \{(x_{2m}^{(1)}, (x_{2m}^{(2)} - x_{2m}^{(1)})/h), \ldots, (x_{2m}^{(m)}, -(x_{2m}^{(m)} + x_{2m}^{(1)})/h), (-x_{2m}^{(1)}, -(x_{2m}^{(2)} - x_{2m}^{(1)})/h), \ldots, (-x_{2m}^{(m)}, (x_{2m}^{(m)} + x_{2m}^{(1)})/h)\}$.

Lemma(Period-2 Stability)

Period-2 is stable if $|x_2^{(1)}| > \left(\frac{h\beta\eta}{2}\right)^{\frac{1}{1-\eta}}$.

Proof

We know that period-2 orbit is stable if

$$\left|\frac{\mathrm{d}\Phi^2(x)}{\mathrm{d}x}\right| = \left|\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right|_{x=-x_2^{(1)}} \left|\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right|_{x=x_2^{(1)}} < 1.$$

Using the relation $\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} = 1 - h\beta\eta x^{\eta-1}$, we obtain

$$0 < (1 - h\beta\eta(x_2^{(1)})^{\eta-1})^2 < 1.$$

This can be reduced to

$$-1 < 1 - h\beta\eta(x_2^{(1)})^{\eta-1} < 1.$$

Using left side inequalities, we obtain $|x_2^{(1)}| > \left(\frac{h\beta\eta}{2}\right)^{\frac{1}{1-\eta}}$ and thus proof is completed.

Lemma(Period-4 Stability)

For the given period-4 points $\{x_4^{(1)}, x_4^{(2)}, -x_4^{(1)}, -x_4^{(2)}\}$, the period-4 is stable if any one of the following conditions satisfy

$$\mathrm{C1)}\ \left|x_4^{(1)}\right| > (h\beta\eta)^{\frac{1}{1-\eta}}, \left|x_4^{(2)}\right| > \left(\frac{h\beta\eta}{1+p_4^1}\right)^{\frac{1}{1-\eta}}$$

$$\mathrm{C2)}\left(\frac{h\beta\eta}{2}\right)^{\frac{1}{1-\eta}} < \left|x_4^{(1)}\right| < (h\beta\eta)^{\frac{1}{1-\eta}}, \left|x_4^{(2)}\right| > \left(\frac{h\beta\eta}{1-p_4^1}\right)^{\frac{1}{1-\eta}}$$

$$\mathrm{C3)}\ \left|x_4^{(1)}\right| < \left(\frac{h\beta\eta}{2}\right)^{\frac{1}{1-\eta}}, \left(\frac{h\beta\eta}{1-p_4^1}\right)^{\frac{1}{1-\eta}} < \left|x_4^{(2)}\right| < \left(\frac{h\beta\eta}{1+p_4^1}\right)^{\frac{1}{1-\eta}}$$

where $p_4^1 = \frac{1}{1-h\beta\eta(x_4^{(1)})^{\eta-1}}$.

Proof

The period-4 is stable if

$$\left|\frac{\mathrm{d}\Phi^4(x)}{\mathrm{d}x}\right| = \left|\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right|_{x=-x_4^{(2)}} \left|\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right|_{x=-x_4^{(1)}} \left|\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right|_{x=x_4^{(2)}} \left|\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x}\right|_{x=x_4^{(1)}} < 1.$$

Using the relation $\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} = 1 - h\beta\eta x^{\eta-1}$, we obtain

$$(1 - h\beta\eta(x_4^{(1)})^{\eta-1})^2(1 - h\beta\eta(x_4^{(2)})^{\eta-1})^2 < 1.$$

This can be rewritten as

$$-1 < (1 - h\beta\eta(x_4^{(1)})^{\eta-1})(1 - h\beta\eta(x_4^{(2)})^{\eta-1}) < 1.$$

We find the different stability conditions for $x_4^{(1)}$ and $x_4^{(2)}$.

Proof

i) $0 < 1 - h\beta\eta(x_4^{(1)})^{\eta-1} < 1$
Dividing by $(1 - h\beta\eta(x_4^{(1)})^{\eta-1})$ on both the sides, it gives

$$\frac{-1}{1 - h\beta\eta(x_4^{(1)})^{\eta-1}} < 1 - h\beta\eta(x_4^{(2)})^{\eta-1} < \frac{1}{1 - h\beta\eta(x_4^{(1)})^{\eta-1}}.$$

From $0 < 1 - h\beta\eta(x_4^{(1)})^{\eta-1} < 1$, we obtain $\left|x_4^{(1)}\right| > (h\beta\eta)^{\frac{1}{1-\eta}}$. Note that $\frac{1}{1-h\beta\eta(x_4^{(1)})^{\eta-1}} = p_4^1 \in (1, \infty)$. Using this in the left inequality, we write

$$\left|x_4^{(2)}\right| > \left(\frac{h\beta\eta}{1 + p_4^1}\right)^{\frac{1}{1-\eta}}.$$

Similarly, it can be shown other cases.

Main Result

Theorem

The system $\Phi(x_1)$ shows only period-2 motion in steady-state for all sampling period.

Remark

- The proposed discrete TSM results only period-2 motion while the direct discretization continuous-time TSM may not result period-2 for all sampling period.
- Desired steady-state bounds can be obtained by choosing suitable sampling period.

Proof

Consider the Lyapunov function $V(k) = x_1^2(k)$. The stability is guaranteed if and only if $\Delta V(k) = V(k+1) - V(k) < 0$ for all $k \in \mathbb{Z}_{\geq 0}$. So,

$$\Delta V(k) = \Delta x_1(k)(2x_1(k) + \Delta x_1(k)) < 0.$$

We have $\Delta x_1(k) = x_1(k+1) - x_1(k) = -h\beta x_1^{\eta}(k)$, so we can write

$$2x_1(k) + \Delta x_1(k) = 2x_1(k) - h\beta x_1^{\eta}(k).$$

Now, we consider the three region as

$$\Omega = \left\{ x_1(k) \ : \ |x_1(k)| \leq \left(\frac{h\beta}{2}\right)^{\frac{1}{1-\eta}} \right\}$$

$$\partial\Omega = \left\{ x_1(k) \ : \ |x_1(k)| = \left(\frac{h\beta}{2}\right)^{\frac{1}{1-\eta}} \right\}$$

$$\Omega_0 = \{ x_1(k) \ : \ x_1(k) = 0 \}$$

proof

It can be verified that

- $\Delta x_1(k) < 0$ and $2x_1(k) + \Delta x_1(k) > 0$ for all $x_1(k) > 0$ and $x_1(k) \notin \Omega$
- $\Delta x_1(k) > 0$ and $2x_1(k) + \Delta x_1(k) < 0$ for all $x_1(k) < 0$ and $x_1(k) \notin \Omega$. This implies $V(k+1) < V(k)$, i.e., the region Ω is *attractive*.
- For all $x_1(k) \in \partial\Omega$, it follows

$$x_1(k+1) = \mp \left(\frac{h\beta}{2}\right)^{\frac{1}{1-\eta}},$$

  this means $x_1(k) \in \partial\Omega$ and $\partial\Omega$ is a *positively invariant* set.

- Similarly consider $x_1(k) \in \Omega \backslash (\partial\Omega \cup \Omega_0)$. So, for $x_1(k) = \pm\alpha \left(\frac{h\beta}{2}\right)^{\frac{1}{1-\eta}}$ with $\alpha \in (0,1)$, we obtain

$$x_1(k+1) = \mp(2-\alpha^{1-\eta})\alpha^{\eta} \left(\frac{h\beta}{2}\right)^{\frac{1}{1-\eta}}$$

  The quantity $(2-\alpha^{1-\eta})\alpha^{\eta-1}$ is always greater than one for $\alpha \in (0,1)$, so the trajectories in very next sampling instant trajectory goes to the opposite side with magnitude higher than the previous instant. Eventually reaches $\partial\Omega$.

Main Result

proof

The period-2 discrete points can be calculated by $\Phi(x_2^{(1)}) = -x_2^{(1)}$. So

$$x_2^{(1)} - h\beta(x_2^{(1)})^{\eta} = -x_2^{(1)}$$

and then, we obtain

$$x_2^{(1)} = \left(\frac{h\beta}{2}\right)^{\frac{1}{1-\eta}}.$$

Therefore, the period-2 motion occurs in the limit set $\{(h\beta/2)^{\frac{1}{1-\eta}}, -(h\beta/2)^{\frac{1}{1-\eta}}\}$.

- It can be seen that the steady-state points satisfy period-2 stability conditions and only period-2 motion occurs
- No periodic orbits occurs other than period-2 since there is no other periodic points

This completes the proof.

Thank You