IEEE IES Distinguished Lecture

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International Summer School on Sliding Mode Control- Variable Structure Systems
State University of Rio de Janeiro (UERJ), Rio de Janeiro, Brazil
April 8-12, 2019

Output Feedback Control-A Multirate Approach

Formulation of Problem

Given a system

$$\dot{x}(t) = Ax(t) + Bu$$
$$y = Cx \tag{1}$$

where $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{m\times 1}$, $y \in \mathbb{R}^{l\times 1}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{l\times n}$ are the state vector, control input, output, system matrix, input matrix and output matrix respectively.

Discrete-time Control

- Give the system required degree of stability.
- Implementation of control is generally by computer (now a days).
- If computer being used in control implementation, closed loop is discrete-time.
- Better use discrete-time control rather than continuous time control.

Controller - Options

State Feedback Based Control

For given (state space representation) system, state feedback based control would give best performance.
Problem:-Availability of all state information, which is not possible always.

Static Output Feedback

Good: Output is available.
Problem : No guarantee of success for a controllable and observable system.

Dynamic Output Feedback

Good: Output + Stability Guaranteed.
Problem : More dynamics in closed loop.

My Option:-Multirate Output Feedback

Controller input (or system output) and controller output (control input) are sampled at different rates.

Multirate Output Feedback

Periodic Output Feedback

Control input sampled faster than System output.

Fast Output Sampling

System output sampled faster than control input.

Periodic Output Feedback Control Law

Plant described by equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2}$$

- (A, B) controllable and (A, C) observable.
- Output sampled at a rate of τ sec.
- Input applied at a rate $\Delta = \frac{\tau}{N}$ sec.

Let us consider the Δ system

$$x(k+1) = \Phi x(k) + \Gamma u(k) \tag{3}$$

Let

$$\begin{aligned}
u(0) &= k_0 y(0) \quad 0 \leq t < \Delta \\
u(\Delta) &= k_1 y(0) \quad \Delta \leq t < 2\Delta \\
u(2\Delta) &= k_2 y(0) \quad 2\Delta \leq t < 3\Delta \\
&\vdots \\
u(\tau - \Delta) &= k_{N-1} y(0) \quad \tau - \Delta \leq t < \tau
\end{aligned} \tag{4}$$

$$\begin{aligned}
u(\tau) &= k_0 y(\tau) \quad \tau \leq t < \tau + \Delta \\
u(\tau + \Delta) &= k_1 y(\tau) \quad \tau + \Delta \leq t < \tau + 2\Delta \\
u(\tau + 2\Delta) &= k_2 y(\tau) \quad \tau + 2\Delta \leq t < \tau + 3\Delta \\
&\vdots \\
u(2\tau - \Delta) &= k_{N-1} y(\tau) \quad 2\tau - \Delta \leq t < 2\tau
\end{aligned} \tag{5}$$

If the control (4) is applied to (3)

$$x(\Delta) = \Phi x(0) + \Gamma k_0 C x(0) = (\Phi + \Gamma k_0 C) x(0) \tag{6}$$

Similarly

$$\begin{aligned} x(2\Delta) &= \Phi x(\Delta) + \Gamma k_1 C x(0) \\ &= \Phi(\Phi + \Gamma k_1 C) x(0) + \Gamma k_1 C x(0) \\ &= \Phi^2 x(0) + \Phi \Gamma k_0 C x(0) + \Gamma k_1 C x(0) \end{aligned} \tag{7}$$

$$\begin{aligned} x(3\Delta) &= \Phi x(2\Delta) + \Gamma k_2 C x(0) \\ &= \Phi^3 x(0) + \Phi^2 \Gamma k_0 C x(0) + \Phi \Gamma k_1 C x(0) + \Gamma k_2 C x(0) \end{aligned} \tag{8}$$

Like above it can be shown

$$x(N\Delta) = \Phi^N x(0) + \Phi^{N-1}\Gamma k_0 Cx(0) + \Phi^{N-2}\Gamma k_1 Cx(0) + \cdots + \Gamma k_{N-1} Cx(0)$$
$$= \Phi^N x(0) + \begin{bmatrix} \Phi^{N-1}\Gamma & \Phi^{N-2}\Gamma & \cdots & \Gamma \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_{N-1} \end{bmatrix} Cx(0) \quad (9)$$

So

$$x(\tau) = (\Phi^N + \Gamma \mathrm{KC})\mathrm{x}(0) \quad (10)$$

where $\begin{bmatrix} \Phi^{N-1}\Gamma & \Phi^{N-2}\Gamma & \cdots & \Gamma \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_{N-1} \end{bmatrix} = \Gamma \mathrm{K}$

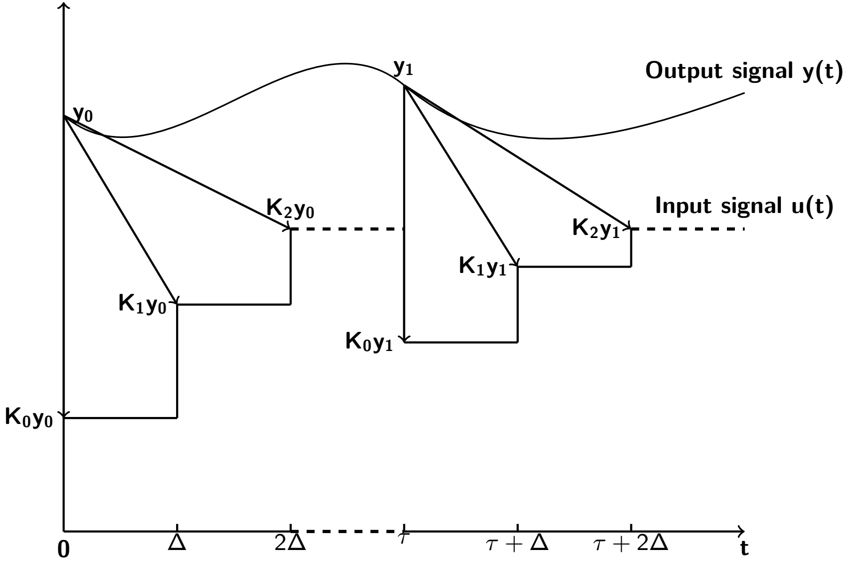


Figure: Pictorial representation of POF control law

Similarly it can be shown

$$x(2\tau) = (\Phi^N + \Gamma \mathrm{KC})\mathrm{x}(\tau) \tag{11}$$

In general the closed loop τ system under the feedback control (4), (5)$\cdots$ becomes

$$x(k+1)\tau = (\Phi_\tau + \Gamma \mathrm{KC})\mathrm{x}(\mathrm{k}\tau) \tag{12}$$

Now let us consider the τ system

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) \tag{13}$$

Let G be the output injection gain such that

$$x(k+1) = (\Phi_\tau + GC)x(k) \tag{14}$$

is stable. This is always possible because (Φ_τ, C) pair is observable.

Periodic Output Feedback Control Law Deduction Cont...

Solution for K

- Solution for K can be found by solving $\Gamma \mathrm{K} = \mathrm{G}$.
- $N \geq$ controllability index, is a sufficient condition for the existence of the solution for K.

Periodic Output Feedback LMI Formulation

- Solving $\Gamma \mathrm{K} = \mathrm{G}$ may give a gain that is high in magnitude, amplifying noise in practical system.
- Hence impose condition $\|K\| < \rho_2$ for noise reduction and $\|\Gamma K - G\| < \rho_1$ for stability during controller design.
- Restrictions posed as LMI problem

$$\begin{bmatrix} -\rho_1^2 I & (\Gamma K - G) \\ (\Gamma K - G)^T & I \end{bmatrix} < 0, \quad \begin{bmatrix} -\rho_2^2 I & K \\ K^T & I \end{bmatrix} < 0 \tag{15}$$

Some Literature:-The concept of multirate output feedback (output faster than input) is quite old.

- G. M. Kranc, "Input-output analysis of multirate feedback systems", IEEE Trans. Auto. Contr., Vol. 3, No. 1, pp. 21-28, Nov. 1957.
- E. Jury, "A note on multirate sampled data systems", IRE Trans. Auto. Contr., Vol. 12, No. 3, pp. 319-320, Jun. 1967.
- P. T. Kabamba, "Control of linear systems using generalized sampled data hold functions", IEEE Trans. Auto. Contr. , Vol. 32, No. 9, pp. 772-783, Sept. 1987.
- T. Hagiwara and M. Araki, "Design of a stable state feedback controller based on multirate sampling of plant output", IEEE Trans. Auto. Contr., Vol. 33, No. 9, pp. 812-819, Sept. 1988.
- H. Werner and K. Furuta, "Simultaneous stabilization based on output measurement", Kybernetika, Vol. 31, pp. 395-411, 1995.

Visualization

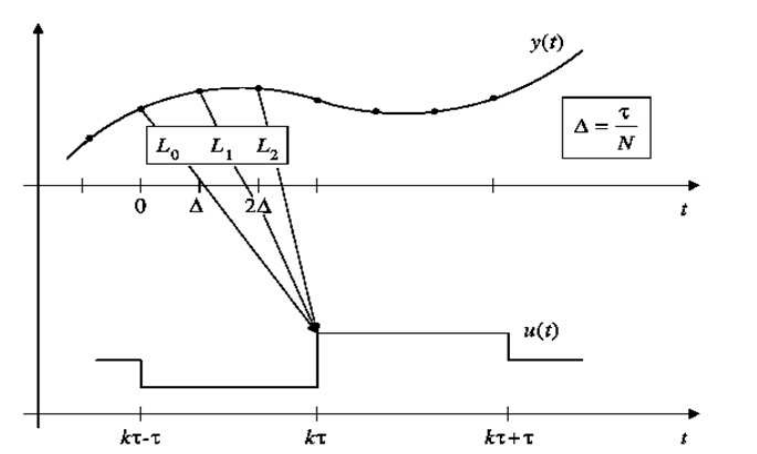


Figure: Multirate Output Feedback

A Multirate Output Sampled System

Computations of the fast output sampling gain

Let us consider the Δ system

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{16}$$

Output is sampled several times in one input sampling i.e., output is sampled after every Δ sec and input is applied at every τ sec. Thus

$$\begin{aligned} y(0) &= Cx(0) \\ y(\Delta) &= Cx(\Delta) = C\Phi x(0) + C\Gamma u(0) && (17) \\ y(2\Delta) &= Cx(2\Delta) = C\Phi x(\Delta) + C\Gamma u(\Delta) \\ &= C\Phi x(\Delta) + C\Gamma u(\Delta) \\ &= C\Phi[\Phi x(0) + \Gamma u(0)] + C\Gamma u(0) \\ &= C\Phi^2 x(0) + C\Phi\Gamma u(0) + C\Gamma u(0) && (18) \end{aligned}$$

A Multirate Output Sampled System

$$
\begin{aligned}
y(3\Delta) &= Cx(3\Delta) = C\Phi x(2\Delta) + C\Gamma u(2\Delta) \\
&= C\Phi^3 x(0) + C\Phi^2\Gamma u(0) + C\Phi\Gamma u(0) + C\Gamma u(0) \qquad (19) \\
&\vdots \\
y(\tau - \Delta) &= C\Phi^{N-1} x(0) + C\Phi^{N-2}\Gamma u(0) + C\Phi^{N-3}\Gamma u(0) + \cdots + C\Gamma u(0) \qquad (20)
\end{aligned}
$$

By staking the output one gets

$$
\begin{aligned}
y(0) &= Cx(0) \\
y(\Delta) &= Cx(\Delta) = C\Phi x(0) + C\Gamma u(0) \\
y(2\Delta) &= C\Phi^2 x(0) + C\Phi\Gamma u(0) + C\Gamma u(0) \\
&\vdots \\
y(\tau - \Delta) &= C\Phi^{N-1} x(0) + C\Phi^{N-2}\Gamma u(0) + C\Phi^{N-3}\Gamma u(0) + \cdots + C\Gamma u(0) \qquad (21)
\end{aligned}
$$

Further

$$\begin{bmatrix} y(0) \\ y(\Delta) \\ y(2\Delta) \\ \vdots \\ y(\tau-\Delta) \end{bmatrix} = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{N-1} \end{bmatrix} x(0) + \begin{bmatrix} 0 \\ C\Gamma \\ C\Phi\Gamma + C\Gamma \\ \vdots \\ \sum_{i=0}^{N-2} C\Phi^i\Gamma \end{bmatrix} u(0) \tag{22}$$

$$y_\tau = C_0 x(0) + D_0 u(0) \tag{23}$$

Similarly it can be shown

$$y_{(k+1)\tau} = C_0 x(k\tau) + D_0 u(k\tau) \tag{24}$$

where $C_0 = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{N-1} \end{bmatrix}$, $D_0 = \begin{bmatrix} 0 \\ C\Gamma \\ C\Phi\Gamma + C\Gamma \\ \vdots \\ \sum_{i=0}^{N-2} C\Phi^i\Gamma \end{bmatrix}$ and $y_{k\tau} = \begin{bmatrix} y(k-1)\tau \\ y(k-1)\tau + \Delta \\ y(k-1)\tau + 2\Delta \\ \vdots \\ y(k\tau - \Delta) \end{bmatrix}$.

A Multirate Output Sampled System

So a fast output sampling feedback system with lifted output becomes

$$\begin{aligned} x(k+1) &= \Phi_\tau x(k) + \Gamma_\tau u(k) \\ y_{k+1} &= C_0 x(k) + D_0 u(k) \end{aligned} \tag{25}$$

Let $u(k) = Fx(k)$ be a stabilizing control for the system (25). Also let us assume $u(k) = [L_0 \quad L_1 \quad \cdots \quad L_{N-1}]y_k$ be the fast output sampling controller. Thus

$$x(k+1) = (\Phi_\tau + \Gamma_\tau F)x(k) \tag{26}$$

$$y_{k+1} = (C_0 + D_0 F)x(k) \tag{27}$$

Further

$$\begin{aligned} y_{k+1} &= (C_0 + D_0 F)(\Phi_\tau + \Gamma_\tau F)^{-1} x(k+1) \\ y_k &= \mathbf{C} x(k) \end{aligned} \tag{28}$$

where $\mathbf{C} = (C_0 + D_0 F)(\Phi_\tau + \Gamma_\tau F)^{-1}$.

A Multirate Output Sampled System

C is invertible if $(\Phi_\tau + \Gamma_\tau F)$ does not have a pole at origin and Δ system is observable and that $N \geq$ observability index of the Δ system.

Now feedback system under fast output sampling feedback becomes

$$x(k+1) = (\Phi_\tau + \Gamma_\tau LC)x(k) \tag{29}$$

Equating (26) and (29)

$$L\mathbf{C} = F \tag{30}$$

Solution for L can be obtain by solving (30) and its solution is guaranteed if C is invertible or it has full rank.

What about the control for $t = 0$?

- For $0 \leq t < \tau, \;\; u(k) = Fx(0)$.
- For $t > \tau, \;\; u(k) = Ly_k$.

Initial State Estimation Consideration

- Let initial control signal $u(0) = Fx(0)$.
- $x(0)$ is assumed and then the control is computed as $u(0) = Fx(0)$.
- As $x(0)$ is not known exactly, there is an error in control signal at $t = 0$. This error propagates through the system and generates an error dynamics.

let the error in control signal be

$$\Delta u(k) = u(k) - Fx(k) \tag{31}$$

Now

$$\begin{bmatrix} x(k) \\ \Delta u(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} x(k+1) \\ \Delta u(k+1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} \tag{33}$$

A Multirate Output Sampled System

Also

$$\begin{bmatrix} x(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} \Phi_\tau & \Gamma_\tau \\ LC_0 & LD_0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \quad (34)$$

Substituting (34) in (33)

$$\begin{bmatrix} x(k+1) \\ \Delta u(k+1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} \Phi_\tau & \Gamma_\tau \\ LC_0 & LD_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix}^{-1} \begin{bmatrix} x(k) \\ \Delta u(k) \end{bmatrix} \quad (35)$$

After Simplification one can write

$$\begin{bmatrix} x(k+1) \\ \Delta u(k+1) \end{bmatrix} = \begin{bmatrix} \Phi_\tau + \Gamma_\tau F & \Gamma_\tau \\ 0 & LD_0 - F\Gamma_\tau \end{bmatrix} \begin{bmatrix} x(k) \\ \Delta u(k) \end{bmatrix} \quad (36)$$

A Multirate Output Sampled System

- The dynamics in (36) is stable provided $\Psi(L) = LD_0 - F\Gamma_\tau$ is stable.
- F can be so selected that $LD_0 - F\Gamma_\tau$ and $(\Phi_\tau + \Gamma_\tau F)$ are stable.
- For faster decay eigenvalues should be close to zero.
- Constraint can be incorporated in design of L as $\|\Psi(L)\| < \rho_2$.

Noise Considerations

L derived from $L\mathbf{C} = F$

- May give control signal of small magnitude even if L elements are large in magnitude $\rightarrow$ **Theoretically.**
- Does not imply good control as **Noise Amplification** is imminent in practical case.

Therefore, Constraint to be considered while designing L is $\|L\| < \rho$.

LMI Formulation of the Design Problem

Approximate Solution $\rightarrow L\mathbf{C} \approx F$

- Helps satisfying above constraints $\|L\mathbf{C} - F\| < \rho_3$.
- Effect on stability is minimal.

A Multirate Output Sampled System

Constraints posed as LMI problem as

$$\begin{bmatrix} -\rho_1^2 I & L \\ L^T & -I \end{bmatrix} < 0, \quad \begin{bmatrix} -\rho_2^2 I & \Psi(L) \\ \Psi^T(L) & -I \end{bmatrix} < 0, \quad \begin{bmatrix} -\rho_3^2 I & LC - F \\ (LC-F)^T & -I \end{bmatrix} < 0 \qquad (37)$$

- Choose a sampling time Δ sec, at which Output is sampled and $\tau = N\Delta$ sec during which the Input is held constant.
- Control signal $u(t)$ for $k\tau < t \leq (k+1)\tau$ is calculated as linear combination of **past** N Outputs.
- Controller Gains are adjusted so as to take care of
  - Noise amplification.
  - Initial Control Error.
  - Closeness to desired State Feedback.

Modified Multirate Output Feedback Technique

What if control is not a linear state feedback ?
Improved Algorithm : Multirate Output Feedback

Consider the output equation

$$y_{k+1} = C_0 x(k) + D_0 u(k) \tag{38}$$

From this the state can be computed as

$$\begin{aligned} C_0^T y_{k+1} &= C_0^T (C_0 x(k) + D_0 u(k)) \\ x(k) &= (C_0^T C_0)^{-1} C_0^T (y_{k+1} - D_0 u(k)) \end{aligned} \tag{39}$$

Substituting this in $x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k)$, we obtain

$$\begin{aligned} x(k+1) &= \Phi_\tau (C_0^T C_0)^{-1} C_0^T y_{k+1} + \left(\Gamma_\tau - \Phi_\tau (C_0^T C_0)^{-1} C_0^T D_0\right) u(k) \\ \Rightarrow x(k) &= \Phi_\tau (C_0^T C_0)^{-1} C_0^T y_k + \left(\Gamma_\tau - \Phi_\tau (C_0^T C_0)^{-1} C_0^T D_0\right) u(k-1) \end{aligned} \tag{40}$$

This computed state can be used for implementing any state feedback based control.

Visualization

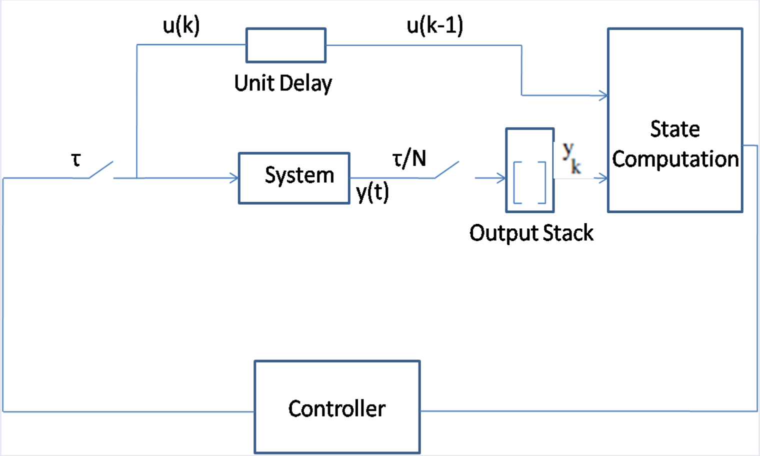


Figure: Improved Multirate Output Feedback

- Output feedback based.
- Inherent control computation time. Useful for fast systems.
- As compared to FOS
  - No restriction on F. In fact, no restriction on $u = Fx$.
  - Initial control error does not propagate.

Dicrete-time Sliding Mode Control with Fast Output Sampling

Review

State Feedback

- Guarantees closed loop system stability
- Requires all the state information
- Observer based feedback control if state measurements are not available

Static Output Feedback

- Easy to implement
- Output feedback do not guarantee closed loop system stability in general
- Observer based control increases complexity

Multirate Output Feedback

- Guarantees closed loop system stability
- Only output measurements are required
- Outputs are sampled and inputs are given to the plant at different rate

Multitrate Output Feedback Based Sliding Mode With Gao's Reaching Law

System Description

Consider a discrete-time system

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + D_\tau f(k)$$
$$y(k) = Cx(k)$$

where $f(k)$ is the uncertainty such that $f(k) = 0$ for $k < 0$ and it satisfies matching condition with system input. Choose the sliding manifold as

$$s(k) = c^\top x(k) = 0$$

where $c^\top \Gamma_\tau \neq 0$. We define $\tilde{d}(k) = c^\top D_\tau f(k)$.

Assumptions

- (Φ_τ, Γ_τ) is a controllable pair.
- (Φ_τ, C) is an observable pair.
- The disturbance $\tilde{d}(k)$ is bounded i.e., $d_l \leq \tilde{d}(k) \leq d_u$.

Define $d_0 = 0.5(d_u + d_l)$ and $d_1 = 0.5(d_u - d_l)$.

Gao's Reaching Law: Review

The reaching law is given as

$$s(k+1) = s(k) - q\tau s(k) - \epsilon\tau \mathrm{sign}(s(k)) + \tilde{d}(k) - d_0 - d_1 \mathrm{sign}(s(k))$$

where $q, \tau, \epsilon > 0$ and $0 < 1 - q\tau < 1$.

Control Design

The sliding dynamics

$$s(k+1) = c^\top \Phi_\tau x(k) + c^\top \Gamma_\tau u(k) + c^\top D_\tau f(k).$$

The control law designed using Gao's reaching law is

$$u(k) = -(c^\top \Gamma_\tau)^{-1} \left(c^\top \Phi_\tau x(k) + (1 - q\tau)s(k) + \epsilon\tau \mathrm{sign}(s(k)) + d_0 + d_1 \mathrm{sign}(s(k)) \right)$$

- It needs all the state information

Fast Output Sampling Technique

Let the system

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + D_\tau f(k)$$
$$y(k) = Cx(k)$$

be sampled at $\Delta = \tau/N$ where N being greater than the observability index of the system. The Δ system can be represented as

$$x(k+1) = \Phi x(k) + \Gamma u(k) + Df(k)$$

where the relation between Δ and τ system is given as

$$\Phi_\tau = \Phi^N, \quad \Gamma_\tau = \sum_{i=0}^{N-1} \Phi^i \Gamma \quad \text{and} \quad D = \left(\sum_{i=0}^{N-1} \Phi^i \right)^{-1} D_\tau .$$

Since the output is sampled fast, the complete system dynamics can be represented as

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + D_\tau f(k)$$
$$y_{k+1} = C_0 x(k) + D_0 u(k) + C_d f(k)$$

where $y_k = [y((k-1)\tau)\ y((k-1)\tau+\Delta) \ldots y(k\tau-\Delta)]^\top$

Fast Output Sampling Technique

$$C_0 = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{N-1} \end{bmatrix} \quad D_0 = \begin{bmatrix} 0 \\ C\Gamma \\ C(\Phi\Gamma + \Gamma) \\ \vdots \\ C\sum_{i=0}^{N-2} \Phi^i \Gamma \end{bmatrix} \quad \text{and} \quad C_d = \begin{bmatrix} 0 \\ C \\ C(\Phi + I) \\ \vdots \\ C\sum_{i=0}^{N-2} \Phi^i \end{bmatrix} D.$$

Now, it can be written

$$C_0^\top y_{k+1} = C_0^\top (C_0 x(k) + D_0 u(k) + C_d f(k)).$$

Since N is greater than observability index so,

$$x(k) = (C_0^\top C_0)^{-1} C_0^\top (y_{k+1} - D_0 u(k) - C_d f(k))$$
$$x(k+1) = L_y y_{k+1} + L_u u(k) + L_d f(k).$$

where $L_y = \Phi_\tau (C_0^\top C_0)^{-1} C_0^\top$, $L_u = \Gamma_\tau - L_y D_0$ and $L_d = I - L_y C_d$. Thus, we obtain

$$x(k) = L_y y_k + L_u u(k-1) + L_d f(k-1).$$

Modified Gao's Reaching Law

- Control input appears in the following relation

$$x(k) = L_y y_k + L_u u(k-1) + L_d f(k-1)$$

- So, the control law need to designed from a modified Gao's reaching law
- This new reaching law would determine new bound for QSM

Modified Gao's Reaching Law

The new reaching law is given as

$$s(k+1) = s(k) - q\tau s(k) - \epsilon\tau \text{sign}(s(k)) + \tilde{g}(k-1) + \tilde{d}(k) - d_0 - g_0 - (d_1 + g_1)\text{sign}(s(k$$

where $\tilde{g}(k) = \left(c^\top \Phi_\tau - c^\top + q\tau c^\top\right) L_d f(k)$ with $g_l \leq \tilde{g}(k) \leq g_u$ and

$$g_0 = 0.5(g_u + g_l) \quad \text{and} \quad g_1 = 0.5(g_u - g_l).$$

Control Design

The state feedback control obtained from the modified reaching law

$$u(k) = -(c^\top \Gamma_\tau)^{-1} \Big[\left(c^\top \Phi_\tau - c^\top + q\tau c^\top \right) x(k) - \tilde{g}(k-1) + d_0 + g_0 + (d_1 + g_1 + \epsilon\tau)\mathrm{sign}(s(k)) \Big]$$

Substituting $x(k) = L_y y_k + L_u u(k-1) + L_d f(k-1)$, it can be easily shown that

$$u(k) = F_y y_k + F_u u(k-1) - (c^\top \Gamma_\tau)^{-1} \left(d_0 + g_0 + (d_1 + g_1 + \epsilon\tau)\mathrm{sign}(s(k)) \right)$$

where

$$F_y = -(c^\top \Gamma_\tau)^{-1} \left(c^\top \Phi_\tau - c^\top + q\tau c^\top \right) L_y$$
$$F_u = -(c^\top \Gamma_\tau)^{-1} \left(c^\top \Phi_\tau - c^\top + q\tau c^\top \right) L_u$$

- This control depends only on past output and control inputs
- But, $s(k) = c^\top x(k)$ still depends on the state $x(k)$

Computation of $\mathrm{sign}(s(k))$

- Since $x(k)$ depends on the uncertainty, replace $\mathrm{sign}(s(k))$ by $\mathrm{sign}(\bar{s}(k))$, where

$$\begin{aligned}\bar{s}(k) &= c^\top L_y y_k + c^\top L_u u(k-1) + l_0 \\ &= s(k) - l(k-1) + l_0\end{aligned}$$

Here, $l(k) = c^\top L_d f(k)$ with bounds as $l_l \leq l(k) \leq l_u$ and

$$l_0 = 0.5(l_u + l_l) \quad \text{and} \quad l_1 = 0.5(l_u - l_l).$$

- Clearly, if $|s(k)| > l_1$, then $\mathrm{sign}(s(k)) = \mathrm{sign}(\bar{s}(k))$.

- So, QSMB can be reached using $\mathrm{sign}(\bar{s}(k))$ correctly if QSMB is greater than l_1.

DTSM Control Design

Computation of $\mathrm{sign}(s(k))$

- In order to obtain QSMB, find

$$\bar{s}(k+1) = (1-q\tau)\bar{s}(k) + (1-q\tau)(l(k-1)-l_0) - (l(k)-l_0) \\ - \epsilon\tau\mathrm{sign}(\bar{s}(k)) + (\tilde{g}(k-1)-g_0) + (\tilde{d}(k)-d_0) - (d_1+g_1)\mathrm{sign}(\bar{s}(k))$$

- Recall that QSMB is the region where $\mathrm{sign}(\bar{s}(k+1)) = -\mathrm{sign}(\bar{s}(k))$. Thus, for $\bar{s}(k) > 0$ and $\bar{s}(k+1) < 0$, one has

$$(1-q\tau)\bar{s}(k) + (1-q\tau)(l(k-1)-l_0) - (l(k)-l_0) \\ - \epsilon\tau + (\tilde{g}(k-1)-g_0) + (\tilde{d}(k)-d_0) - (d_1+g_1) < 0.$$

- Now using $\bar{s}(k) = s(k) - l(k-1) + l_0$ in the above, yield

$$(1-q\tau)s(k) - (l(k)-l_0) - \epsilon\tau + (\tilde{g}(k-1)-g_0) + (\tilde{d}(k)-d_0) - (d_1+g_1) < 0.$$

Computation of $\mathrm{sign}(s(k))$

- Assuming $s(k) = \delta_y$, we see that

$$\delta_y < \frac{\epsilon\tau + l_1 + 2(d_1 + g_1)}{1 - q\tau}.$$

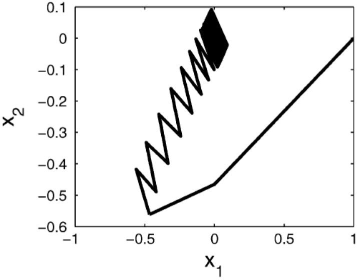
- Observe that $\delta_y > l_1$ due to $(1 - q\tau) < 1$. Thus, outside QSMB the $\mathrm{sign}(\bar{s}(k)) = \mathrm{sign}(s(k))$.

Numerical Example

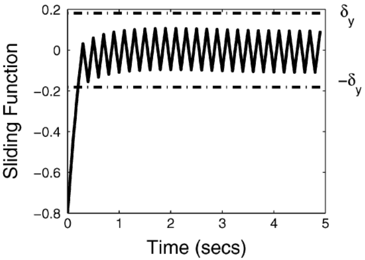
Second Order System

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \sin(0.1k)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k).$$

The different design variables given as $c^\top = [-0.8\ 1]$, $d_0 = 0$, $d_1 = 0.01$, $g_0 = 0$, $g_1 = 0.0086$, $l_0 = 0$, $l_1 = 0.0105$, $q = 2$, $\epsilon = 1$ and $N = 2$.



(a)



(b)

Multitrate Output Feedback Based Sliding Mode With Bartoszewicz Reaching Law

System Description

Consider a discrete-time system

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + \tilde{d}(k)$$
$$y(k) = Cx(k)$$

where $\tilde{d}(k)$ is the uncertainty such that $\tilde{d}(k) = 0$ for $k < 0$ and it satisfies matching condition with system input. Choose the sliding manifold as

$$s(k) = c^\top x(k) = 0$$

where $c^\top \Gamma_\tau \neq 0$. We define $d(k) = c^\top \tilde{d}(k)$.

Assumptions

- (Φ_τ, Γ_τ) is a controllable pair.
- (Φ_τ, C) is an observable pair.
- The disturbance $d(k)$ is bounded i.e., $d_l \leq d(k) \leq d_u$.

Define $d_0 = 0.5(d_u + d_l)$ and $\delta_d = 0.5(d_u - d_l)$.

Bartoszewicz Reaching Law

Bartoszewicz proposed reaching law as

$$s(k+1) = d(k) - d_0 + s_d(k+1)$$

where $s_d(k)$ is a priori known function such that the following apply:

1) if $|s(0)| > 2\delta_d$, then

$$\begin{aligned} s_d(0) &= s(0) \\ s_d(k)\cdot s(0) &\geq 0, \quad \text{for any} \quad k \geq 0 \\ s_d(k) &= 0, \quad \text{for any} \quad k \geq k^* \\ |s_d(k+1)| &< |s_d(k)| - 2\delta_d, \quad \text{for any} \quad k < k^* \end{aligned}$$

2) Otherwise, i.e., if $|s(0)| \leq 2\delta_d$, then $s_d(k) = 0$ for any $k \geq 0$.

The positive integer k^* is chosen to achieve a tradeoff between faster convergence and magnitude of control u. One of possible definition of $s_d(k)$ when $|s(0)| \geq 2\delta_d$ is

$$s_d(k) = \frac{k^* - k}{k^*} s(0) \quad k = 0, 1, \ldots, k^*$$

$$k^* \leq \frac{|s(0)|}{2\delta_d}.$$

Fast Output Sampling Technique

Let the system

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + \tilde{d}(k)$$
$$y(k) = Cx(k)$$

be sampled at $\Delta = \tau/N$ where N being greater than the observability index of the system. The Δ system can be represented as

$$x(k+1) = \Phi x(k) + \Gamma u(k) + d'(k)$$

where the relation between Δ and τ system is given as

$$\Phi_\tau = \Phi^N, \quad \Gamma_\tau = \sum_{i=0}^{N-1} \Phi^i \Gamma \quad \text{and} \quad d'(k) = \left(\sum_{i=0}^{N-1} \Phi^i \right)^{-1} \tilde{d}(k).$$

Since the output is sampled fast, the complete system dynamics can be represented as

$$x(k+1) = \Phi_\tau x(k) + \Gamma_\tau u(k) + \tilde{d}(k)$$
$$y_{k+1} = C_0 x(k) + D_0 u(k) + C_d \tilde{d}(k)$$

Fast Output Sampling Technique

$$C_0 = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{N-1} \end{bmatrix} \quad D_0 = \begin{bmatrix} 0 \\ C\Gamma \\ C(\Phi\Gamma + \Gamma) \\ \vdots \\ C\sum_{i=0}^{N-2} \Phi^i \Gamma \end{bmatrix} \quad \text{and} \quad C_d = \begin{bmatrix} 0 \\ C \\ C(\Phi + I) \\ \vdots \\ C\sum_{i=0}^{N-2} \Phi^i \end{bmatrix} \left(\sum_{i=0}^{N-1} \Phi^i \right)^{-1}$$

Now, it can be written

$$C_0^\top y_{k+1} = C_0^\top (C_0 x(k) + D_0 u(k) + C_d \tilde{d}(k)).$$

Since N is greater than observability index so,

$$x(k) = (C_0^\top C_0)^{-1} C_0^\top (y_{k+1} - D_0 u(k) - C_d \tilde{d}(k))$$
$$x(k+1) = L_y y_{k+1} + L_u u(k) + L_d \tilde{d}(k).$$

where $L_y = \Phi_\tau (C_0^\top C_0)^{-1} C_0^\top$, $L_u = \Gamma_\tau - \Phi_\tau (C_0^\top C_0)^{-1} C_0^\top D_0$ and $L_d = I - \Gamma_\tau (C_0^\top C_0)^{-1} C_0^\top C_d$. Thus, we obtain

$$x(k) = L_y y_k + L_u u(k-1) + L_d \tilde{d}(k-1).$$

Remarks

- The states of the system depend only on the immediate past output information
- Thus, in the control law consists of output information while guaranteeing closed loop system stability

Modified Reaching Law

Denote $e(k) = c^\top \Phi_\tau L_d \tilde{d}(k)$. Then, $e(k)$ can be bounded as $e_l \leq e(k) \leq e_u$. Further, define mean and spread of $e(k)$ as

$$e_0 = 0.5(e_l + e_u) \quad \text{and} \quad \delta_e = 0.5(e_u - e_l).$$

Now, the new reaching law based on the output information given as

$$s(k+1) = d(k) - d_0 + e(k-1) - e_0 + s_d(k+1)$$

where $s_d(k)$ is a priori function and satisfies Bartoszewicz reaching law conditions for the disturbance bound $\delta_d + \delta_e$ instead of δ_d. The function $s_d(k)$ can be given as

$$s_d(k) = \frac{k^* - k}{k^*} s(0) \quad k = 0, 1, \ldots, k^*$$

$$k^* \leq \frac{|s(0)|}{\ldots}, \quad s_d(k) = 0 \quad k > k^*.$$

Control Law Design

The control which satisfies modified reaching law can be derived as

$$\begin{aligned} s(k+1) &= c^\top(\Phi_\tau x(k) + \Gamma_\tau u(k) + \tilde{d}(k)) \\ &= c^\top \Phi_\tau x(k) + c^\top \Gamma_\tau u(k) + d(k). \end{aligned}$$

Thus, the control law is proposed as

$$u(k) = -(c^\top\Gamma_\tau)^{-1}(c^\top\Phi_\tau x(k) + d_0 - e(k-1) + e_0 - s_d(k+1)).$$

This control law expression contains $x(k)$, so replace this by output information as derived earlier. Then, it follows

$$\begin{aligned} u(k) &= -(c^\top\Gamma_\tau)^{-1}(c^\top\Phi_\tau[L_y y_k + L_u u(k-1) + L_d\tilde{d}(k-1)] + d_0 - e(k-1) + e_0 - s_d(k + \\ &= -(c^\top\Gamma_\tau)^{-1}(c^\top\Phi_\tau L_y y_k + c^\top\Phi_\tau L_u u(k-1) + e(k-1) + d_0 - e(k-1) + e_0 - s_d(k \\ &= -(c^\top\Gamma_\tau)^{-1}(c^\top\Phi_\tau L_y y_k + c^\top\Phi_\tau L_u u(k-1) + d_0 + e_0 - s_d(k+1)). \end{aligned}$$

Remark

- At time instant $k = 0$, there is no past output information available so, the control expression can be obtained by ignoring $e(k-1)$ and e_0 as

$$u(0) = -(c^\top \Gamma_\tau)^{-1}(c^\top \Phi_\tau x(0) + d_0 - s_d(1))$$

It can be noticed that for $k > \max\{k^*, 2\}$, we have $s_d(k) = 0$. So,

$$s(k) = d(k-1) - d_0 + e(k-2) - e_0.$$

The band size can be obtained as

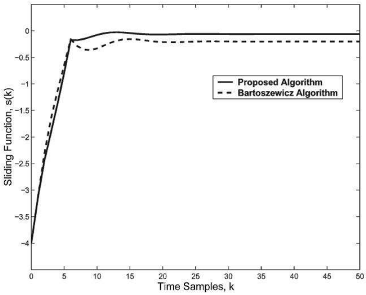
$$\begin{aligned} |s(k)| &= |d(k-1) - d_0 + e(k-2) - e_0| \\ &\leq |d(k-1) - d_0| + |e(k-2) - e_0| \\ &\leq \delta_d + \delta_e. \end{aligned}$$

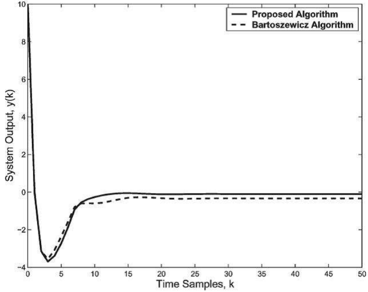
Second Order System

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0.4 & -0.3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ \sin(k/2)\exp(-k/5) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k).$$

The different design variables given as $c^\top = [-0.4\ 1]$, $d_u = 0.564$, $d_l = -0.162$, $\delta_d = 0.363$, $e_u = 0.1131$, $d_l = -0.395$, $\delta_d = 0.254$ and $k^* = 6$.





Thank You