Note that this is a sinusoidal pattern in a two-dimensional (ξ, α) domain, with the amplitude $\sqrt{x_0^2 + y_0^2}$ and the phase $\psi = \arctan(y_0/x_0)$. Of course, the Radon transform is periodic in α with 2π. Projections for $0 \leq \alpha < \pi$ are sufficient to calculate all transform values. Thus, a point in the (x, y) domain transforms to a sinusoidal pattern in Radon transform domain. This relation can be used to transform a sinusoidal pattern into a point via inverse Radon transform. □

It may be shown that the Fourier transform of the Radon transform, along direction $\xi = x\cos(\alpha) + y\sin(\alpha)$, is the two-dimensional Fourier transform of the original signal $f(x,y)$ along the same direction in transform domain. Thus, by knowing all the projections, for $0 \leq \alpha < \pi$, we can calculate the two-dimensional Fourier transform of $f(x,y)$. It means that we can reconstruct a two-dimensional function $f(x,y)$ from its projections or integrals (basic theorem for computed tomography).

In the chapters that follow, we will often use a specific form of the two-dimensional function $P(t, \Omega)$, with time and frequency as independent variables. Then, for the two-dimensional Fourier transform we will use the notation

$$A(\theta, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(t, \Omega) e^{-j\theta t + j\Omega\tau} dt d\Omega. \tag{1.180}$$

Note that this is indeed the inverse Fourier transform over Ω and the Fourier transform over t. However, we can still use all the discrete calculation routines, for the standard two-dimensional Fourier transform, since $2\pi A(-\theta, \tau)$ is the standard two-dimensional transform. Thus, in the discrete domain, one axis should be reversed and the result multiplied by the number of samples N.

The inverse two-dimensional Fourier transform, in this notation, reads

$$P(t, \Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\theta, \tau) e^{j\theta t - j\Omega\tau} d\theta d\tau. \tag{1.181}$$

It can be calculated by using the standard two-dimensional Fourier transform routines, by appropriate reversal and multiplication, as well.

1.5 PROBLEMS

Problem 1.1. If a signal $x(t)$ is periodic with period T, find the period of signal $y(t) = x(3t/4)x(t/3 - T)$.

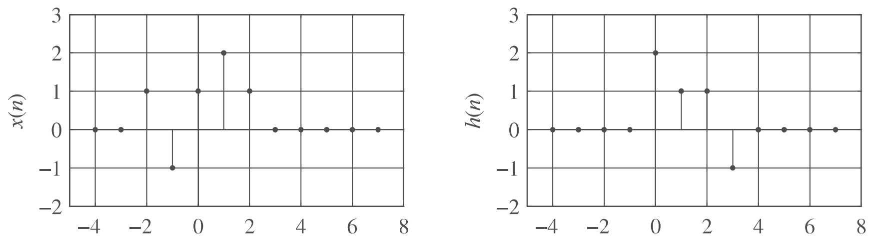

Figure 1.14 Signals for Problem 1.6.

Problem 1.2. Signal $x(t)$ is periodic with period 2. Its nonzero Fourier series coefficients are $X_0 = 5$, $X_1 = 3j$, $X_2 = 2$. Signal $h(t)$ is periodic with period 4. Its nonzero coefficients are $H_0 = 4$, $H_1 = 1 + j$, $H_{-1} = 1 - j$. Check the periodicity and find the Fourier series coefficients of signal $y(t) = x(t) + h(t)$.

Problem 1.3. Find the output signal of an LTI system, with the impulse response $h(t) = te^{-t}u(t)$, to the input signal $x(t) = u(t)$.

Problem 1.4. Find the Fourier transform of a signal $x(t)$ defined by

$$x(t) = \begin{cases} \sin(\pi t) & \text{for } |t| < 1 \\ 0 & \text{for } |t| \geq 1 \end{cases}.$$

Problem 1.5. The Fourier transform of a time-limited signal $x(t)$ ($x(t) = 0$ for $|t| > 4$) is

$$X(\Omega) = 4\pi j \frac{\sin(2\Omega)}{4\Omega^2 - \pi^2}.$$

Find the Fourier series coefficients for signals $y(t)$ and $z(t)$ obtained as periodic extensions of $x(t)$ with periods $T = 4$ and $T = 8$, respectively.

Problem 1.6. Calculate discrete-time convolution of signals $x(n)$ and $h(n)$ shown in Fig. 1.14.

Problem 1.7. Consider three causal linear time-invariant systems in cascade. Impulse responses of these systems are $h_1(n)$, $h_2(n)$, and $h_2(n)$, respectively. The impulse response of the second and the third system is $h_2(n) = u(n) - u(n-2)$, while the impulse response of the whole system,

$$h(n) = h_1(n) *_n h_2(n) *_n h_2(n),$$

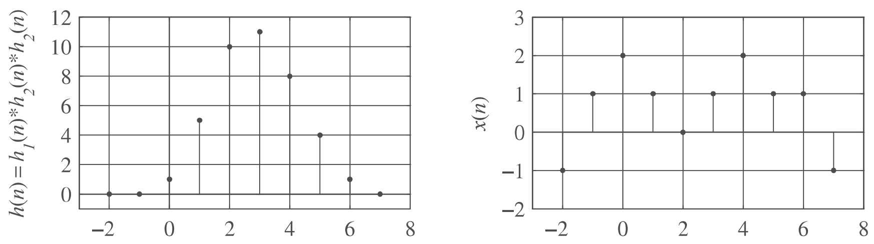

Figure 1.15 Problem 1.7, impulse response $h(n)$ (left) and Problem 1.9, discrete signal $x(n)$ (right).

is shown in Fig. 1.15 (left).

Find $h_1(n)$ and $y(n) = h(n) *_n x(n)$, with $x(n) = \delta(n) - \delta(n-1)$.

Problem 1.8. Find the Fourier transform of the following discrete-time signal (triangular window)

$$w_T(n) = \left(1 - \frac{|n|}{N+1}\right)[u(n+N) - u(n-N-1)].$$

with N being an even number.

Problem 1.9. The discrete-time signal $x(n)$ is given in Fig. 1.15 (right). Without calculating its Fourier transform $X(e^{j\omega})$ find

$$X(e^{j0}), \quad X(e^{j\pi}), \quad \int_{-\pi}^{\pi} X(e^{j\omega})d\omega, \quad \int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega,$$

and a signal whose Fourier transform is the real part of $X(e^{j\omega})$, denoted by $\mathrm{Re}\{X(e^{j\omega})\}$.

Problem 1.10. Impulse response of a discrete system is given as:

$$\text{(a)} \quad h(n) = \frac{\sin(n\pi/3)}{n\pi},$$

$$\text{(b)} \quad h(n) = \frac{\sin^2(n\pi/3)}{(n\pi)^2},$$

$$\text{(c)} \quad h(n) = \frac{\sin((n-2)\pi/4)}{(n-2)\pi}.$$

Find the responses to $x(n) = \sin(n\pi/6)$.

Problem 1.11. Find the value of integral

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin^2((N+1)\omega/2)}{\sin^2(\omega/2)} d\omega.$$

Problem 1.12. The Fourier transform of a continuous signal $x(t)$ is different from 0 only within $3\Omega_1 < \Omega < 5\Omega_1$. Find the maximum possible sampling interval Δt such that the signal can be reconstructed based on the samples $x(n\Delta t)$.

Problem 1.13. Sampling of a signal is done twice, with the sampling interval $\Delta t = 2\pi/\Omega_m$ that is twice larger than the sampling interval required by the sampling theorem ($\Delta t = \pi/\Omega_m$ is required). After first sampling process, the discrete-time signal $x_1(n) = \Delta t x(n\Delta t)$ is formed, while after the second sampling process $x_2(n) = \Delta t x(n\Delta t + a)$ is formed. Show that we can reconstruct continuous signal $x(t)$ based on $x_1(n)$ and $x_2(n)$ if $a \neq k\Delta t$, that is, if samples $x_1(n)$ and $x_2(n)$ do not overlap in continuous-time.

Problem 1.14. In general, a sinusoidal signal $x(t) = A\sin(\Omega_0 t + \varphi)$ is described with three parameters A, Ω_0 and φ. Thus, generally speaking, three points of $x(t)$ would be sufficient to find three signal parameters. If we know the signal $x(t)$ at $t = t_0$, $t = t_0 + \Delta t$ and $t = t_0 - \Delta t$ what is the relation to reconstruct, for example, Ω_0, which is usually the most important parameter of a sinusoid?

Problem 1.15. What is the relation between $X(k)$ and $X(N-k)$ for real-valued signals? If $X(k)$ is real-valued, what should the relation be between $x(n)$ and $x(N-n)$?

Problem 1.16. The relationship between the argument in the DFT and the continuous signal frequency is given by

$$\Omega = \begin{cases} \frac{2\pi k}{N\Delta t} & \text{for } 0 \leq k \leq N/2-1 \\ \frac{2\pi(k-N)}{N\Delta t} & \text{for } N/2 \leq k \leq N-1. \end{cases}$$

This is achieved by using shift functions in programs. Show that the shift will not be necessary if we use the signal $x(n)(-1)^n$. The order of DFT values will be starting from the lowest negative frequency, toward the highest positive frequency.

Problem 1.17. In order to illustrate the algorithms for the fast DFT calculation (the FFT algorithms), show that a DFT of N elements can be split into two DFTs of $N/2$ elements, by a simple splitting of the original signal of N samples into two parts of $N/2$ samples. Use this property to prove that the calculation savings can be achieved in the DFT calculation.

Problem 1.18. Find the DFT of signal $x(n) = \exp(j4\pi\sqrt{3}n/N)$, with $N = 16$. If the signal is interpolated four times, find the displacement bin and compare it with the true frequency value. What is the displacement bin if the general formula is applied without interpolation?

Problem 1.19. The random variable $\varepsilon(n)$ is stationary and Cauchy distributed with probability density function

$$p_{\varepsilon(n)}(\xi) = \frac{a}{1+\xi^2}.$$

Find the coefficient a, mean, and variance.

Problem 1.20. Consider a linear time-invariant system whose input is

$$x(n) = w(n)u(n)$$

and the impulse response is

$$h(n) = a^n u(n),$$

where $w(n)$ is a stationary real-valued noise with mean μ_w and auto-correlation $r_{ww}(n,m) = \sigma_w^2\delta(n-m) + \mu_w^2$. Find the mean and the variance of the output signal.

Problem 1.21. Find the mean, auto-correlation, and spectral power density of the random signal

$$x(n) = w(n) + \sum_{k=1}^{N} a_k e^{j(\omega_k n + \theta_k)},$$

where $w(n)$ is a stationary real-valued noise with mean μ_w and auto-correlation $r_{ww}(n,m) = \sigma_w^2\delta(n-m) + \mu_w^2$ and θ_k are random variables uniformly distributed over $-\pi < \theta_k \leq \pi$. All random variables are statistically independent.