Frequency-Response Masking Approach for the Synthesis of Sharp Linear Phase Digital Filters

YONG CHING LIM, MEMBER, IEEE

Abstract —If each delay element of a linear phase low-pass digital filter is replaced by M delay elements, an $(M+1)$-band filter is produced. The transition-width of this $(M+1)$-band filter is $1/M$ that of the prototype low-pass filter. A complementary filter can be obtained by subtracting the output of the $(M+1)$-band filter from a suitably delayed version of the input. The complementary filter is an $(M+1)$-band filter whose passbands and stopbands are the stopbands and passbands, respectively, of the original $(M+1)$-band filter. If the frequency responses of the original $(M+1)$-band filter and its complementary filter are properly masked and recombined, narrow transition-band filter can be obtained. This technique can be used to design sharp low-pass, high-pass, bandpass, and bandstop filters with arbitrary passband bandwidth.

I. Introduction

LINEAR PHASE FIR digital filters have many advantages such as guaranteed stability, free of phase distortion, and low coefficient sensitivity [1], [2]. A serious disadvantage of FIR filters is its complexity. This problem becomes particularly acute in sharp filters. Fig. 1 shows a filter length versus transition width plot [3], [4] for minimax optimum low-pass filters with 0.2-dB peak-to-peak bassband ripple and 40-dB stopband attenuation. As can be seen from Fig. 1, the filter length is inversely proportional to transition-width and the complexity becomes prohibitively high for sharp filters.

Several methods have been proposed in the literature for reducing the complexity of sharp FIR filters [5]–[13]. In multirate filtering [8]–[12], the input signal's bandwidth is first reduced by using a moderate transition-width narrow-band filter. The sampling rate is then reduced and the output signal is further filtered at the reduced sampling rate. Finally, the sampling rate can be restored by interpolation. In recursive running sum prefilter implementation [5], [6], a large number of evenly spaced zeros are placed on the unit circle. The zero at dc is then eliminated by a dc pole. Interpolating the impulse response of a low-pass filter has the effect of reducing its passband width and transition width by the interpolation ratio. The interpolated impulse response [7] approach makes use of this property to synthesize sharp filters. The above methods are very attractive for narrow-band filtering. They can also be used to synthesize wide-band filters by subtracting the narrow-band output from the delayed version of the input signal. The bandwidth of the resulting wide-band filter can again be reduced using the interpolated impulse-response method [13].

Manuscript received August 20, 1985; revised October 15, 1985.
The author is with the Department of Electrical Engineering, National University of Singapore, Singapore, 0511.
IEEE Log Number 8407132.

In this paper, we present a new method which uses a frequency-response masking technique for implementing sharp filters with arbitrary bandwidth. The resulting filter has very sparse coefficients. For a given frequency-response specification, its effective filter length (including both zero and nonzero coefficients) is only slightly longer than the infinite wordlength minimax optimum filter meeting the same specification. Since only a very small fraction of its coefficient values are nonzero, its complexity is very much lower than the infinite wordlength minimax optimum filter. When the frequency-response masking technique is used together with the multiplierless design method, the complexity of the resulting filter is reduced to a minimum.

This paper is divided into seven sections. The usefulness and shortcoming of a simple frequency response masking approach are presented in Section II. Our new technique is based upon the masking of the frequency responses of complementary filters. Its underlying principle is presented in Section III and design details are presented in Sections IV–VII.

II. Frequency Response Masking: Narrow-Band Filter Design

In this section, we present the principle of frequency response masking, its usefulness and shortcoming for implementing narrow-band filters.

Consider a low-pass filter with z-transform transfer function $H_a(z)$, transition width Δ_a, and frequency response $H_a(e^{j\omega})$ as shown in Fig. 2(a). Replacing each delay of this filter by M delays, a filter with z-transform transfer function $H_b(z)=H_a(z^M)$ and frequency response $H_b(e^{j\omega})=H_a(e^{jM\omega})$ as shown in Fig. 2(b) is obtained. If $H_b(e^{j\omega})$ is masked by the frequency response $H_c(e^{j\omega})$ as shown in Fig. 2(c), the resulting frequency response $H_d(e^{j\omega})=H_b(e^{j\omega})H_c(e^{j\omega})$ is shown in Fig. 2(d). The transition width of $H_d(e^{j\omega})$ is Δ_a/M. If $H_b(e^{j\omega})$ is masked by $H_e(e^{j\omega})$ as shown in Fig. 2(e), the resulting frequency response $H_f(e^{j\omega})=H_b(e^{j\omega})H_e(e^{j\omega})$ is shown in Fig. 2(f). The transition width is again Δ_a/M. Fig. 2 illustrates a

0098-4094/86/0400-0357$01.00

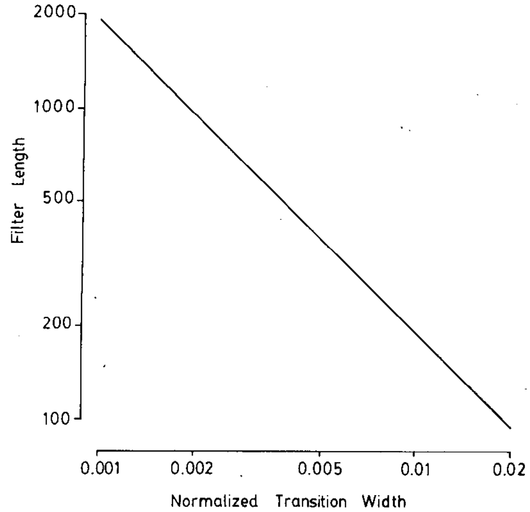

Fig. 1. Filter length versus normalized transition-width plot for an FIR low-pass filter with 0.2-dB peak-to-peak passband ripple and −40-dB stopband ripple. The frequency axis is normalized with respect to sampling frequency.

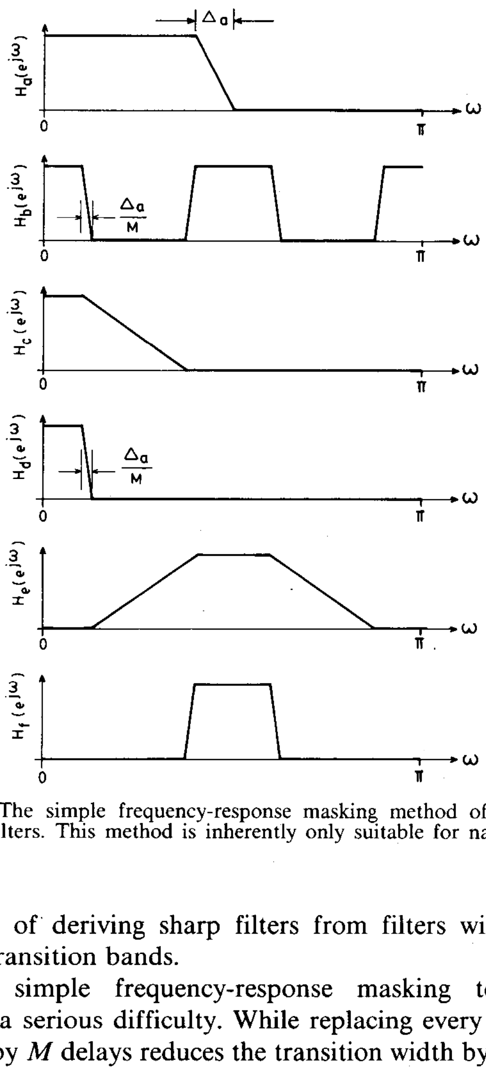

Fig. 2. The simple frequency-response masking method of obtaining sharp filters. This method is inherently only suitable for narrow-band filters.

method of deriving sharp filters from filters with much wider transition bands.

This simple frequency-response masking technique suffers a serious difficulty. While replacing every delay in $H_a(z)$ by M delays reduces the transition width by a factor of M, the passband bandwidth is also reduced by the same factor. Hence, it is suitable only for narrow-band design. It can be shown that the interpolated impulse-response approach [7] is a special case of this frequency-response masking technique where the frequency response of the interpolator belongs to the same category as that of Fig. 2(c).

III. Frequency-Response Masking: Arbitrary Bandwidth Design

Two linear phase filters F_a and F_c are said to be a complementary pair if $|F_a(e^{j\omega})+F_c(e^{j\omega})|=1$, where $F_a(e^{j\omega})$ and $F_c(e^{j\omega})$ are the frequency responses of F_a and F_c, respectively. For a linear phase FIR filter of length N, its frequency response $F_a(e^{j\omega})$ can be expressed as

$$F_a(e^{j\omega}) = e^{-j((N-1)/2)\omega}R(\omega) \tag{1}$$

where $R(\omega)$ is a trigonometric function of ω [1], [2].

The frequency response of the complementary filter is

$$F_c(e^{j\omega}) = e^{-j((N-1)/2)\omega}\{1-R(\omega)\}. \tag{2}$$

If the z-transform transfer function of F_a is $F_a(z)$, then the z-transform transfer function of F_c is given by

$$F_c(z) = z^{-((N-1)/2)} - F_a(z) \tag{3}$$

where F_c can be implemented by subtracting the output of F_a from the delayed version of the input as shown in Fig. 3(a). The extra delays for deriving F_c from F_a need not be implemented explicitly since the delays in F_a can be used for this purpose as shown in Fig. 3(b). Consider a low-pass filter F_a with $R(\omega)$ as shown in Fig. 4(a). The cutoff frequencies are θ and ϕ, respectively. The frequency response of its complementary filter F_c is shown in Fig. 4(b). Two filters F_a' and F_c' are formed by replacing each delay of F_a and F_c by M delays. The frequency responses of F_a' and F_c' for odd N are $F_a'(e^{j\omega})$ and $F_c'(e^{j\omega})$, respectively, as shown in Fig. 4(c) where $F_a'(e^{j\omega})=F_a(e^{jM\omega})$ and $F_c'(e^{j\omega})=F_c(e^{jM\omega})$. Two masking filters F_{Ma} and F_{Mc} with frequency responses $F_{Ma}(e^{j\omega})$ and $F_{Mc}(e^{j\omega})$, as shown in Fig. 4(d), may be used to mask $F_a'(e^{j\omega})$ and $F_c'(e^{j\omega})$, respectively. If the outputs of F_{Ma} and F_{Mc}, are added, as shown in Fig. 5, the frequency response $F(e^{j\omega})$ of the resulting filter F is shown in Fig. 4(e). If ω_p and ω_s are the band edges of F, it can be shown that

$$\omega_p = \frac{2m\pi+\theta}{M} \tag{4a}$$

and

$$\omega_s = \frac{2m\pi+\phi}{M} \tag{4b}$$

where m is an integer less than M. The following points should be noted. First, the group delay of F_{Ma} and that of F_{Mc} must be equal. This means that the length of F_{Ma} and F_{Mc} must either be both odd or both even and that leading delays must be added to either F_{Ma} or F_{Mc} to equalize their group delays if necessary. Second, in order to avoid half sample delay, $(N-1)M$ must be even.

In Fig. 4(e), the frequency response of F near the transition-band is determined mainly by that of F_a'. If the frequency responses of F_{Ma} and F_{Mc} are those shown in Fig. 4(f), the frequency response of F will be that shown in Fig. 4(g) and that the frequency response near the transi-

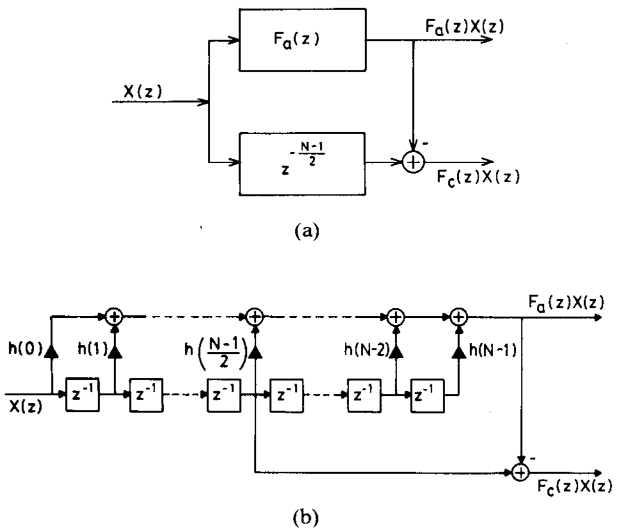

Fig. 3. Realizations of F_c.

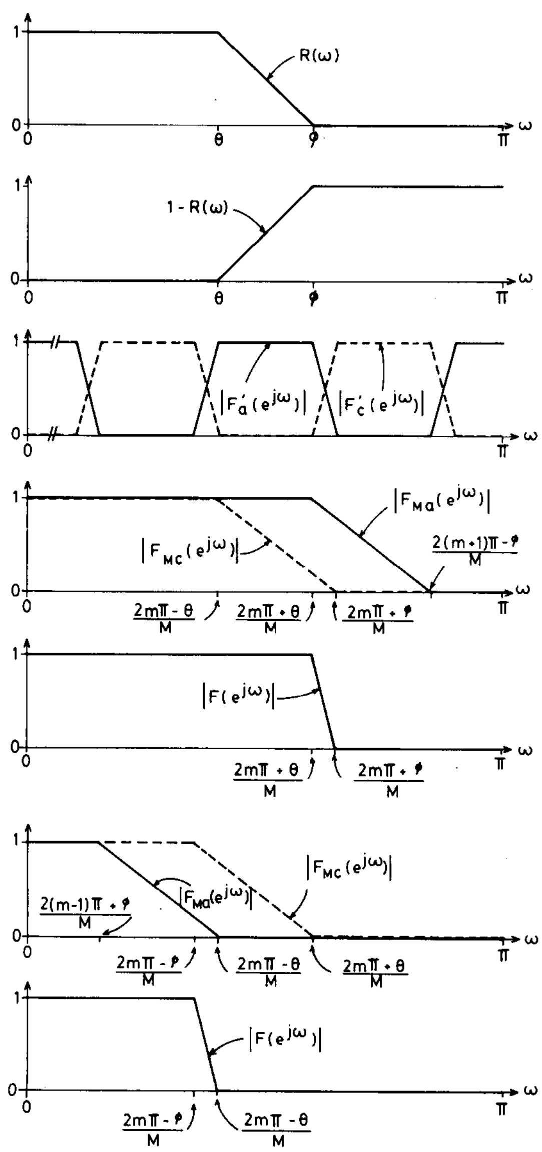

Fig. 4. Frequency-response masking of complementary filters. This method can be used to synthesize wide-band sharp filters.

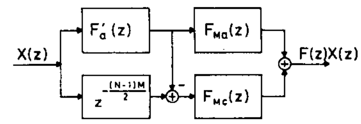

Fig. 5. The structure of a filter synthesized using the frequency-response masking technique.

tion-band will be determined mainly by that of F_c'. It can be shown that in this case the band edges ω_p and ω_s are given by

$$\omega_p = \frac{2m\pi - \phi}{M} \tag{5a}$$

and

$$\omega_s = \frac{2m\pi - \theta}{M}. \tag{5b}$$

In a synthesis problem, ω_p and ω_s are given and m, M, θ, ϕ must be determined. In particular, we wish to choose M such that the overall complexity of the filter is minimized with respect to some criterion. Hence, it is necessary to express θ, ϕ, and m in terms of ω_p, ω_s, and M. To achieve this end, we observe that

$$0 < \theta < \phi < \pi. \tag{6}$$

To ensure that (4a) and (4b) yield a solution with $0 < \theta < \phi$, we have

$$m = \lfloor \omega_p M/(2\pi) \rfloor \tag{4c}$$

$$\theta = \omega_p M - 2m\pi \tag{4d}$$

$$\phi = \omega_s M - 2m\pi \tag{4e}$$

where $\lfloor \omega_p M/(2\pi) \rfloor$ is the largest integer less than $\omega_p M/(2\pi)$. To ensure that (5a) and (5b) yield a solution with $0 < \theta < \phi$, we have

$$m = \lceil \omega_s M/(2\pi) \rceil \tag{5c}$$

$$\theta = 2m\pi - \omega_s M \tag{5d}$$

$$\phi = 2m\pi - \omega_p M \tag{5e}$$

where $\lceil \omega_s M/(2\pi) \rceil$ is the smallest integer larger than $\omega_s M/(2\pi)$. For any set of ω_p, ω_s, and M, only one of (4) or (5) (but not both (4) and (5)) will yield a set of θ and ϕ satisfying the constraint $\phi < \pi$.

Since the transition width of $F_a(e^{j\omega})$ is $M(\omega_s - \omega_p)$, for a given ω_p and ω_s, the transition width of $F_a(e^{j\omega})$ increases with increasing M. Hence, the complexity of F_a decreases with increasing M. The sum of the transition widths of $F_{Ma}(e^{j\omega})$ and $F_{Mc}(e^{j\omega})$ is $1/M$ and decreases with increasing M. Before embarking on a discussion on finding the value of M which gives the minimum overall complexity, we shall examine the effects of $F_a'(e^{j\omega})$, $F_{Ma}(e^{j\omega})$, and $F_{Mc}(e^{j\omega})$ on $F(e^{j\omega})$; this is presented in the next section.

IV. Ripples of $F(e^{j\omega})$

Let $G_a'(\omega)$ and $\delta_a'(\omega)$ be the desired value and deviation, respectively, of $F_a'(e^{j\omega})$, i.e., $G_a'(\omega) = 1$ in the passband of $F_a'(e^{j\omega})$ and $G_a'(\omega) = 0$ in the stopband of $F_a'(e^{j\omega})$. The ripple is $\delta_a'(\omega)$. In the transition band of $F_a'(e^{j\omega})$, we shall define $G_a'(\omega)$ to be equal to $F_a'(e^{j\omega})$ (with an error in the linear phase term), i.e., $\delta_a'(\omega)=0$. Similarly, let $G_{Ma}(\omega)$ and $G_{Mc}(\omega)$, and $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$ be the desired values and deviation, respectively, of $F_{Ma}(e^{j\omega})$ and $F_{Mc}(e^{j\omega})$. If $G(\omega)$ and $\delta(\omega)$ are, respectively, the desired value and deviation of $F(e^{j\omega})$, then we have

$$\begin{aligned} G(\omega)+\delta(\omega) &= \{G_{Ma}(\omega)+\delta_{Ma}(\omega)\}\{G_a'(\omega)+\delta_a'(\omega)\} \\ &\quad + \{G_{Mc}(\omega)+\delta_{Mc}(\omega)\}\{1-G_a'(\omega)-\delta_a'(\omega)\}. \quad (7) \end{aligned}$$

We shall examine the effects of the frequency responses of F_a', F_{Ma}, and F_{Mc} on the frequency response of F in three frequency ranges.

Frequency range 1: $G_{Ma}(\omega) = G_{Mc}(\omega) = 1$. In this frequency range, $G(\omega)=1$, and (7) can be simplified to

$$\begin{aligned} \delta(\omega) &= G_a'(\omega)\{\delta_{Ma}(\omega)-\delta_{Mc}(\omega)\}+\delta_{Mc}(\omega) \\ &\quad + \delta_a'(\omega)\{\delta_{Ma}(\omega)-\delta_{Mc}(\omega)\}. \quad (8) \end{aligned}$$

Ignoring the second-order term, (8) simplifies to

$$\delta(\omega) \approx G_a'(\omega)\{\delta_{Ma}(\omega)-\delta_{Mc}(\omega)\}+\delta_{Mc}(\omega). \quad (9)$$

When

$$G_a'(\omega)=1, \quad \delta(\omega)\approx\delta_{Ma}(\omega). \quad (10a)$$

When

$$G_a'(\omega)=0, \quad \delta(\omega)\approx\delta_{Mc}(\omega). \quad (10b)$$

When

$$0<G_a'(\omega)<1,$$
$$|\delta(\omega)| \leqslant \text{max of } \{|\delta_{Ma}(\omega)|, |\delta_{Mc}(\omega)|\}. \quad (10c)$$

Frequency range 2: $G_{Ma}(\omega) = G_{Ma}(\omega) = 0$. In this frequency range, $G(\omega)=0$, and, after ignoring the second order term, (7) can be simplified to

$$\delta(\omega) \approx G_a'(\omega)\{\delta_{Ma}(\omega)-\delta_{Mc}(\omega)\}+\delta_{Mc}(\omega). \quad (11)$$

When

$$G_a'(\omega)=1, \quad \delta(\omega)\approx\delta_{Ma}(\omega). \quad (12a)$$

When

$$G_a'(\omega)=0, \quad \delta(\omega)\approx\delta_{Mc}(\omega). \quad (12b)$$

When

$$0<G_a'(\omega)<1,$$
$$|\delta(\omega)| \leqslant \text{max of } \{|\delta_{Ma}(\omega)|, |\delta_{Mc}(\omega)|\}. \quad (12c)$$

Frequency range 3: The remaining not covered by frequency ranges 1 and 2. This corresponds to $(2m\pi-\theta)/M<\omega<(2(M+1)\pi-\phi)/M$ for the frequency responses of $G_{Ma}(\omega)$ and $G_{Mc}(\omega)$ shown in Fig. 4(d), and corresponds to $(2(m-1)\pi+\phi)/M<\omega<(2m\pi+\theta)/M$ for $G_{Ma}(\omega)$ and $G_{Mc}(\omega)$ shown in Fig. 4(f). Consider the case where $G_{Ma}(\omega)$ and $G_{Mc}(\omega)$ are shown in Fig. 4(d). For $(2m\pi-\theta)/M<\omega<\omega_p$, $G(\omega)=G_a'(\omega)=G_{Ma}(\omega)=1$. Neglecting the second-order term $\delta_a'(\omega)\delta_{Ma}(\omega)$, we have

$$\delta(\omega) \approx \delta_{Ma}(\omega)+\delta_a'(\omega)\{1-G_{Mc}(\omega)\}. \quad (13a)$$

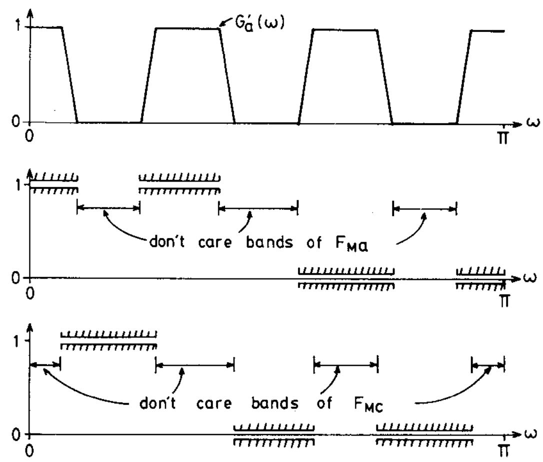

Fig. 6. F_{Ma} and F_{Mc} are low-pass filters with don't care bands within their passbands and stopbands.

TABLE I
THE ESTIMATED LENGTHS OF F_a, F_{Ma}, AND F_{Mc} FOR VARIOUS M; THE CONSTRAINTS ON THE PARITIES OF N, N_{Ma}, AND N_{Mc} ARE NOT CONSIDERED

M	N	N_{Ma}	N_{Mc}	$N+N_{Ma}+N_{Mc}$	Effective length
2	200	1	26	227	424
3	134	5	33	172	432
4	100	13	19	132	415
6	67	19	33	119	429
7	58	18	60	136	459
8	50	100	19	169	492
9	45	41	33	119	437
11	37	64	34	135	460
12	34	93	33	160	489
13	31	30	193	254	583
14	29	53	60	142	452

$G_{Mc}(\omega)$ decreases from unity to zero as ω increases from $(2m\pi-\theta)M$ to ω_p. Hence, for $(2m\pi-\theta)/M<\omega<\omega_p$

$$|\delta(\omega)| \leqslant |\delta_{Ma}(\omega)|+|\delta_a'(\omega)|. \quad (13b)$$

For $\omega_s<\omega<(2(m+1)\pi-\phi)/M$, $G(\omega)=G_a'(\omega)=G_{Mc}(\omega)=0$. Neglecting the second-order term $\delta_a'(\omega)\delta_{Mc}(\omega)$, we have

$$\delta(\omega) \approx G_{Ma}(\omega)\delta_a'(\omega)+\delta_{Mc}(\omega). \quad (14a)$$

$G_{Ma}(\omega)$ decreases from unity to zero as ω increases from ω_s to $(2(m+1)\pi-\phi)/M$. Hence, for $\omega_s<\omega<(2(m+1)\pi-\phi)/M$

$$|\delta(\omega)| \leqslant |\delta_a'(\omega)|+|\delta_{Mc}(\omega)|. \quad (14b)$$

The bounds of (13b) and (14b) are overly pessimistic. $G_a(\omega)$ can be designed in such a way that $\delta_a'(\omega)$ partially compensates for $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$.

Similarly, bounds as those of (13b) and (14b) can be derived for the case of Fig. 4(f).

Remark: In this section we have shown that where $G_{Ma}(\omega)$ and $G_{Mc}(\omega)$ are both equal to zero or one, $\delta(\omega)$ is determined mainly by $\delta_{Ma}(\omega)$ or $\delta_{Mc}(\omega)$ depending on whether $G_a'(\omega)$ is zero or one; the effect of $\delta_a'(\omega)$ is of secondary important. Hence, F_{Ma} and F_{Mc} are lowpass filters with don't care bands within their passbands and stopbands as shown in Fig. 6. These don't care bands help reduce the complexity of F_{Ma} and F_{Mc}. (*Warning*: This will

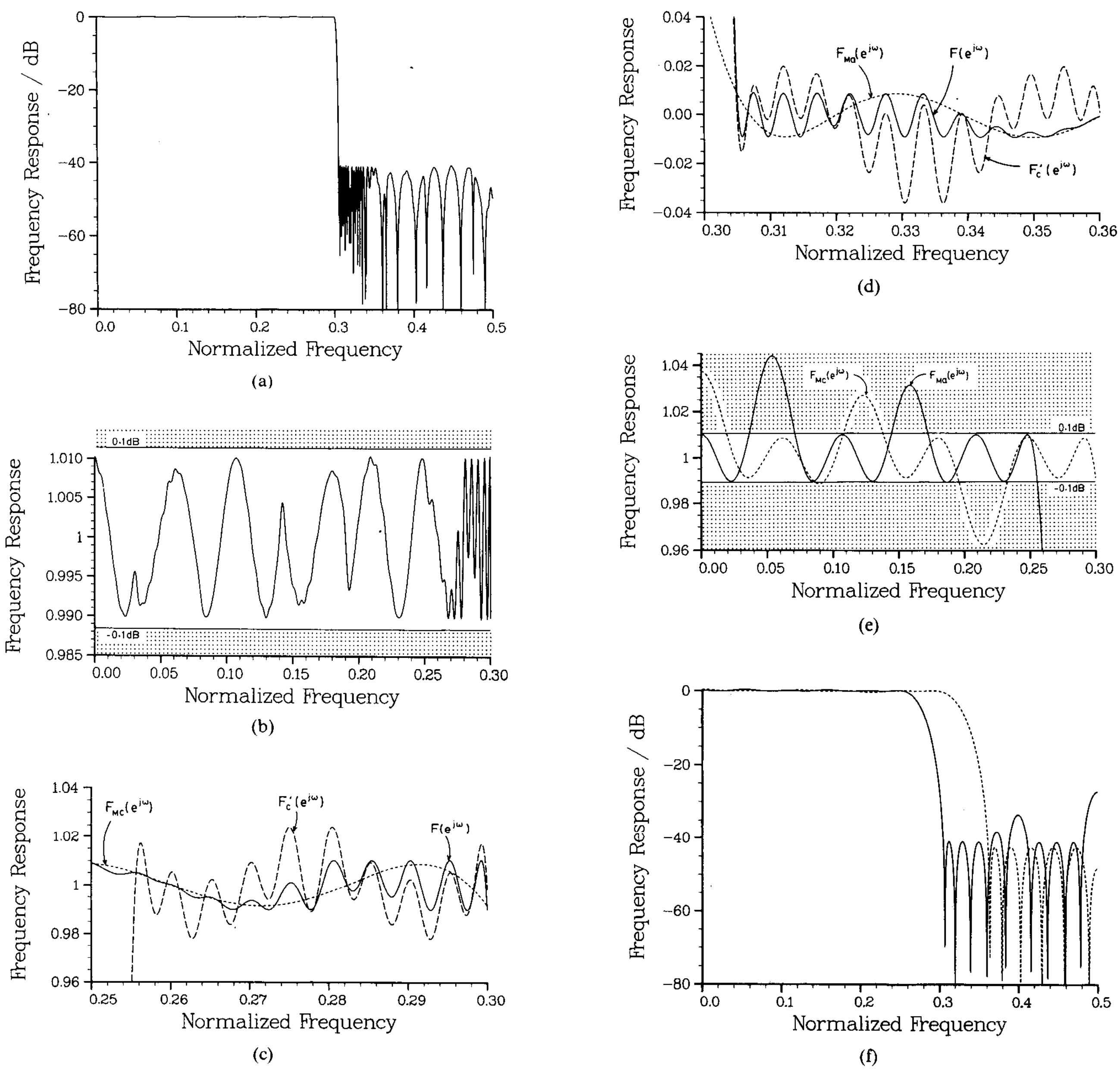

Fig. 7. (a) and (b) show the frequency response of a filter synthesized using frequency response masking technique. The frequency response of the subfilters are shown in (c)–(f). The frequency axes are normalized with respect to sampling frequency. The filter lengths are $N = 45$, $N_{Ma} = 41$, and $N_{Mc} = 33$.

work if δ_{Ma} and δ_{Mc} in the don't care bands are not large. Hence, it is safer to apply a low weighting to δ_{Ma} and δ_{Mc} instead of "don't care" in the don't care bands.) Near the transition-band of F, $G_{Ma}(\omega)$ is not equal to $G_{Mc}(\omega)$; $\delta(\omega)$ is a function of $\delta_a'(\omega)$, $\delta_{Ma}(\omega)$, and $\delta_{Mc}(\omega)$. It is possible to design the filter F_a such that $\delta_a'(\omega)$ partially compensates for $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$.

V. Optimization of $F_a(\omega)$

It is shown in Section IV that it is possible to design $R(\omega)$ so that $\delta_a'(\omega)$ partially compensates for $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$ for those frequencies near the transition-band of F. It should be pointed out that the converse is not true; if $F_a'(\omega)$ is obtained first, it is not possible to derive $F_{Ma}(\omega)$ and $F_{Mc}(\omega)$ to produce significant effects in compensating for $\delta_a'(\omega)$. This is because the effective length of F_a' is $(N-1)M+1$ and is much larger than that of F_{Ma} and F_{Mc}. Hence, F_{Ma} and F_{Mc} should be designed first and then F_a' designed to compensate for $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$. The magnitude of $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$ should be 10 percent–15 percent smaller then the maximum allowable magnitude of $\delta(\omega)$ to give room to the second order error term due to $\delta_a'(\omega)$.

In order to optimize $R(\omega)$ so as to compensate for $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$, a linear equation relating $\delta(\omega)$ and $R(\omega)$ in a useful form must be obtained. This can be

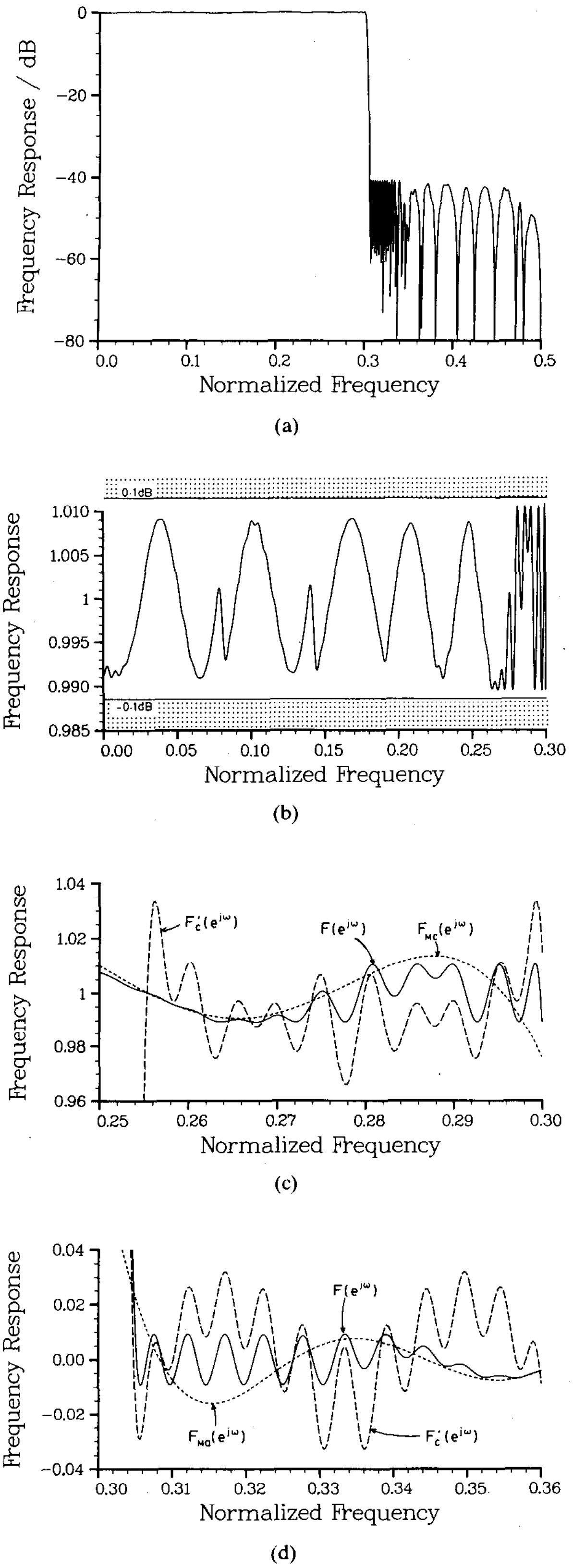

Fig. 8. (a) and (b) show the frequency response of a filter synthesized using frequency-response masking technique. The frequency responses of the subfilters are shown in (c) and (d). The frequency axes are normalized with respect to sampling frequency. The filter lengths are $N = 45$, $N_{Ma} = 38$, and $N_{Mc} = 30$. The constraints on the ripples of $F_{Ma}(e^{j\omega})$ and $F_{Mc}(e^{j\omega})$ near the band edges are slightly relaxed to reduce N_{Ma} and N_{Mc}.

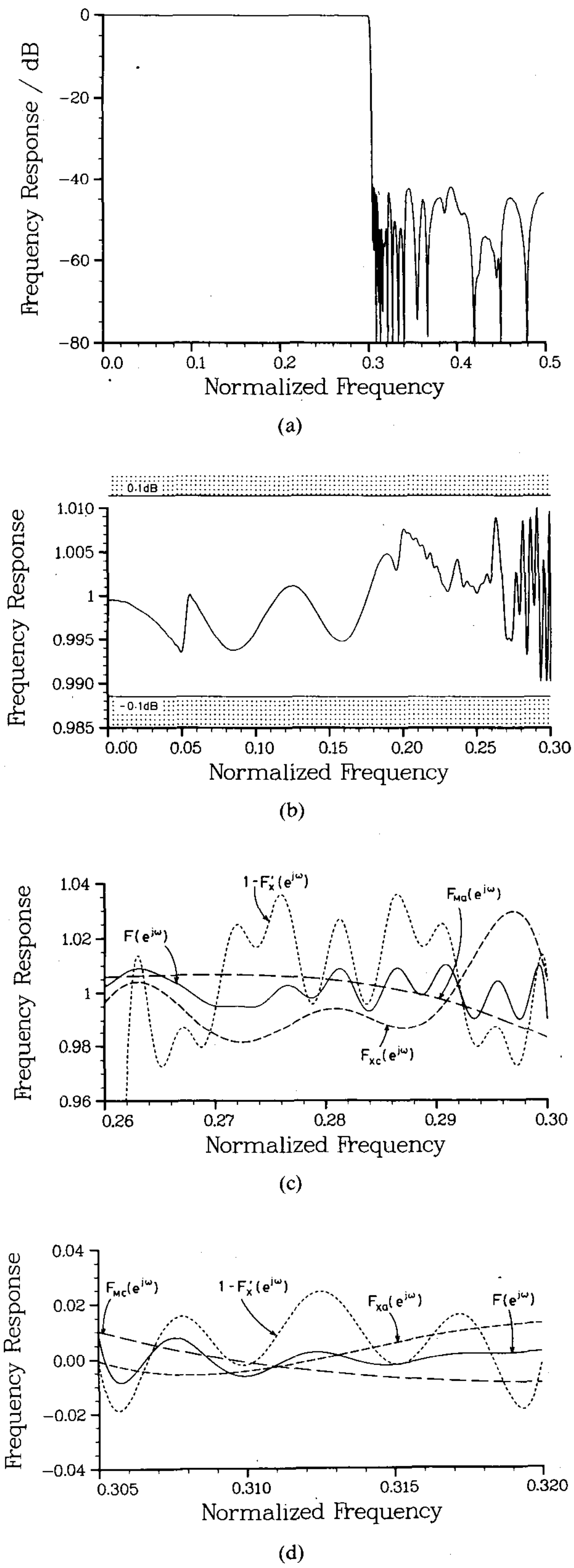

Fig. 9. (a) and (b) show the frequency response of a filter synthesized using frequency-response masking technique. The frequency responses of the subfilters are shown in (c) and (d). The frequency axes are normalized with respect to sampling frequency. The coefficient values are shown in Table II.

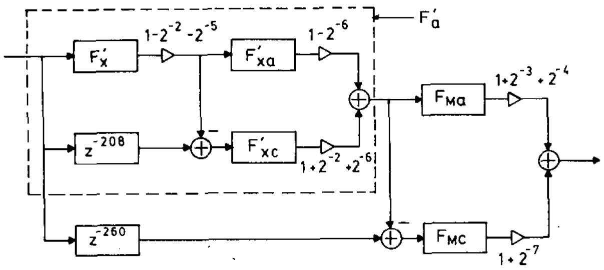

Fig. 10. The structure of a two-level frequency-response masking filter. The subfilter F_a' is synthesized using frequency-response masking technique.

achieved by rearranging (7) to the form of (15)

$$\delta(\omega) = R(M\omega)\{G_{Ma}(\omega) + \delta_{Ma}(\omega) - G_{Mc}(\omega) - \delta_{Mc}(\omega)\} + G_{Mc}(\omega) + \delta_{Mc}(\omega) - G(\omega). \quad (15)$$

Equation (15) should be evaluated on a dense frequency grid covering the frequency range $(2m\pi - \theta)/M < \omega < (2(m+1)\pi - \phi)/M$ for the case corresponding to Fig. 4(d), and covering the frequency range $(2(m-1)\pi + \phi)/M < \omega < (2m\pi + \phi)/M$ for the case corresponding to Fig. 4(f). The minimization of $|\delta(\omega)|$ in (15) is a linear programming filter design problem [14]–[16] and can be solved by a standard mathematical programming package. If finite wordlength coefficient values are needed then finite wordlength filter design methods [17]–[22] may be used. The Reméz exchange algorithm can also be used for the optimization if the desired gain and ripple weighting are properly selected.

VI. Optimization of M

There is no known closed-form analytic expression for finding the optimum M. However, a good choice of M can be obtained by estimating the filter complexity for each M and then selecting the M which corresponds to the lowest estimate. We shall illustrate this method by an example. Consider the design of a linear phase FIR low-pass filter to meet the following specifications:

$$\left.\begin{array}{l}\text{passband frequencies: } 0 \leqslant \omega/2\pi \leqslant 0.3\\ \text{maximum passband ripple: } \pm 0.1\ \text{dB}\\ \text{stopband frequencies: } 0.305 \leqslant \omega/2\pi \leqslant 0.5\\ \text{minimum stopband attenuation: } 40\ \text{dB.}\end{array}\right\} \quad (16)$$

For any given transition bandwidth and passband–stopband ripples, the filter length of the infinite wordlength minimax optimum low-pass filter can be estimated using the formular reported in [3], [4]. To meet the specifications of (16), the estimated length of the infinite wordlength minimax optimum filter is 383.

In the frequency response masking technique, the frequency response ripple magnitudes of F_{Ma} and F_{Mc} must be 10 percent–15 percent less than the allowed ripple magnitude. The ripple magnitude of F_a depends on those of F_{Ma} and F_{Mc}. For the purpose of estimating the filter

TABLE II
Coefficient Values of the Filters of Fig. 10

$h_x(0) = 2^{-8} + 2^{-9}$	$= h_x(416)$
$h_x(16) = -2^{-7} - 2^{-9}$	$= h_x(400)$
$h_x(32) = -2^{-6}$	$= h_x(384)$
$h_x(48) = -2^{-6} + 2^{-9}$	$= h_x(368)$
$h_x(64) = 2^{-7} + 2^{-8}$	$= h_x(352)$
$h_x(80) = 2^{-5} + 2^{-8}$	$= h_x(336)$
$h_x(96) = 2^{-5} + 2^{-10}$	$= h_x(320)$
$h_x(112) = -2^{-6} - 2^{-9}$	$= h_x(304)$
$h_x(128) = -2^{-4} - 2^{-6}$	$= h_x(288)$
$h_x(144) = -2^{-4} - 2^{-7}$	$= h_x(272)$
$h_x(160) = 2^{-6} + 2^{-9}$	$= h_x(256)$
$h_x(176) = 2^{-3} + 2^{-4}$	$= h_x(240)$
$h_x(192) = 2^{-2} + 2^{-3}$	$= h_x(224)$
$h_x(208) = 2^{-1} - 2^{-4}$	
$h_{xc}(0) = 2^{-10}$	$= h_{xc}(104)$
$h_{xc}(4) = -2^{-7} + 2^{-10}$	$= h_{xc}(100)$
$h_{xc}(8) = -2^{-8} - 2^{-9}$	$= h_{xc}(96)$
$h_{xc}(12) = 2^{-8} + 2^{-9}$	$= h_{xc}(92)$
$h_{xc}(16) = 2^{-7} + 2^{-9}$	$= h_{xc}(88)$
$h_{xc}(20) = -2^{-6} + 2^{-9}$	$= h_{xc}(84)$
$h_{xc}(24) = -2^{-6} - 2^{-7}$	$= h_{xc}(80)$
$h_{xc}(28) = 2^{-7} + 2^{-8}$	$= h_{xc}(76)$
$h_{xc}(32) = 2^{-5} + 2^{-7}$	$= h_{xc}(72)$
$h_{xc}(36) = -2^{-6} - 2^{-10}$	$= h_{xc}(68)$
$h_{xc}(40) = -2^{-4} - 2^{-6}$	$= h_{xc}(64)$
$h_{xc}(44) = 2^{-7} + 2^{-8}$	$= h_{xc}(60)$
$h_{xc}(48) = 2^{-2} - 2^{-8}$	$= h_{xc}(56)$
$h_{xc}(52) = 2^{-2} + 2^{-3}$	
$h_{xa}(0) = -2^{-6}$	$= h_{xa}(48)$
$h_{xa}(4) = -2^{-5} + 2^{-8}$	$= h_{xa}(44)$
$h_{xa}(8) = -2^{-7}$	$= h_{xa}(40)$
$h_{xa}(12) = 2^{-5} + 2^{-6}$	$= h_{xa}(36)$
$h_{xa}(16) = 2^{-3} + 2^{-6}$	$= h_{xa}(32)$
$h_{xa}(20) = 2^{-2} - 2^{-6}$	$= h_{xa}(28)$
$h_{xa}(24) = 2^{-2} + 2^{-6}$	
$h_{Ma}(0) = -2^{-8} - 2^{-9}$	$= h_{Ma}(14)$
$h_{Ma}(1) = 2^{-6} + 2^{-9}$	$= h_{Ma}(13)$
$h_{Ma}(2) = -2^{-6} - 2^{-8}$	$= h_{Ma}(12)$
$h_{Ma}(3) = 2^{-8} + 2^{-10}$	$= h_{Ma}(11)$
$h_{Ma}(4) = 2^{-5} + 2^{-6}$	$= h_{Ma}(10)$
$h_{Ma}(5) = -2^{-3} + 2^{-9}$	$= h_{Ma}(9)$
$h_{Ma}(6) = 2^{-3} + 2^{-4}$	$= h_{Ma}(8)$
$h_{Ma}(7) = 2^{-1} + 2^{-3}$	$=$
$h_{Mc}(0) = -2^{-8}$	$= h_{Mc}(22)$
$h_{Mc}(1) = -2^{-8}$	$= h_{Mc}(21)$
$h_{Mc}(2) = 2^{-7} + 2^{-8}$	$= h_{Mc}(20)$
$h_{Mc}(3) = 2^{-8}$	$= h_{Mc}(19)$
$h_{Mc}(4) = -2^{-6} - 2^{-7}$	$= h_{Mc}(18)$
$h_{Mc}(5) = -2^{-7}$	$= h_{Mc}(17)$
$h_{Mc}(6) = 2^{-5} + 2^{-6}$	$= h_{Mc}(16)$
$h_{Mc}(7) = 2^{-7}$	$= h_{Mc}(15)$
$h_{Mc}(8) = -2^{-4} - 2^{-5}$	$= h_{Mc}(14)$
$h_{Mc}(9) = -2^{-7} - 2^{-8}$	$= h_{Mc}(13)$
$h_{Mc}(10) = 2^{-2} + 2^{-4}$	$= h_{Mc}(12)$
$h_{Mc}(11) = 2^{-1} + 2^{-8}$	

complexity, we shall assume that the ripple magnitudes of F_a, F_{Ma}, and F_{Mc} are each 85 percent of the allowed magnitude. Using this assumption, the estimated lengths of F_a, F_{Ma}, and F_{Mc} to meet the specification of (16) are shown in Table I. The parities of N, N_{Ma}, and N_{Mc} are not considered in the tabulation of Table I. The sum $N + N_{Ma}$

$+ N_{Mc}$ gives the total number of nonzero multipliers and can be used as a measure of complexity. This sum is also tabulated in Table I. It is minimum when $M = 6$ or 9. The frequency response of a design with $M = 9$ is shown in Fig. 7. Note the compensation effect of $\delta_a'(\omega)$ in Fig. 7(c) and 7(d). Also note the large ripple magnitudes of $|F_{Ma}(e^{j\omega})|$ and $|F_{Mc}(e^{j\omega})|$ in their respective don't care bands in Fig. 7(e) and Fig. 7(f).

The compensation effect of $\delta_a'(\omega)$ can be further utilized to reduce the complexity of F_{Ma} and F_{Mc} by relaxing the constraints on $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$ near ω_p and ω_s. It should be pointed out that the compensation of $\delta_{Ma}(\omega)$ and $\delta_{Mc}(\omega)$ by $\delta_a'(\omega)$ is effective only when $|F_{Ma}(\omega) - F_{Mc}(\omega)|$ is large, i.e., when $\omega \approx \omega_p$ or $\omega \approx \omega_s$. A design (with $M = 9$) which exploit this ripple compensation effect is shown in Fig. 8. The lengths of F_a, F_{Ma}, and F_{Mc} are 45, 38, and 30, respectively.

VII. Higher Level Masking of Frequency Responses

The frequency-response masking method can be applied to reduce the complexities of the subfilters F_a, F_{Ma}, and F_{Mc}. We shall use the example of Table I with $M = 4$ as an example. In this case the estimated length of F_a is 100. The filter F_a may be implemented as a system of subfilters using the frequency-response masking technique. The complexity of the filter may be further reduced by constraining all the coefficient values to be a sum or difference of two powers-of-two using the powers-of-two design technique [17]–[19]. Fig. 9 shows the frequency response of such a design. The coefficient values are tabulated in Table II. The subfilters are organized as shown in Fig. 10. This filter requires 202 shift–add operations per sampling interval. Comparing with the infinite precision minimax optimum design which requires 383 multiply and 382 add operations per sampling interval, our new technique produces a significant saving in the arithmetic operations.

VIII. Conclusion

A new technique for synthesizing sharp linear phase digital filters is presented. In this technique, the frequency responses of complementary pair filters are masked by the frequency responses of two appropriate masking filters. The outputs of the masking filters are combined to produce the desired output. This technique produces filters with very sparse coefficients and so the resulting filter has very low arithmetic complexity.

References

[1] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975, ch. 3.

[2] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975, ch. 4–5.

[3] L. R. Rabiner, J. H. McClellan, and T. W. Parks, "FIR digital filter design techniques using weighted Chebyshev approximation," *Proc. IEEE*, vol. 64, pp. 595–609, Apr. 1975.

[4] O. Herrmann, L. R. Rabiner, and D. S. K. Chan, "Practical design rules for optimum finite impulse response lowpass digital filters," *Bell Syst. Tech. J.*, vol. 52, pp. 769–799, July–Aug. 1973.

[5] J. W. Adams and A. N. Willson, Jr., "A new approach to FIR digital filters with fewer multipliers and reduced sensitivity," *IEEE Trans. Circuits Syst.*, vol. CAS-30, pp. 277–283, May 1983.

[6] J. W. Adams and A. N. Willson, Jr., "Some efficient digital prefilter structures," *IEEE Trans. Circuits Syst.*, vol. CAS-31, pp. 260–266, Mar. 1984.

[7] Y. Neuvo, C. Y. Dong, and S. K. Mitra, "Interpolated finite impulse response filters," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-32, pp. 563–570, June 1984.

[8] R. E. Crochiere and L. R. Rabiner, "Optimum FIR digital filter implementations for decimation, interpolation, and narrow-band filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 444–456, Oct. 1975.

[9] D. J. Goodman, M. J. Carey, "Nine digital filters for decimation and interpolation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-25, pp. 121–126, Apr. 1977.

[10] M. G. Bellanger, J. L. Daguet and G. P. Lepagnol, "Interpolation, extrapolation, and reduction of computation speed in digiter filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-22, pp. 231–235, Aug. 1974.

[11] R. R. Shively, "On multistage finite impulse response (FIR) filters with decimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 353–357, Aug. 1975.

[12] L. R. Rabiner and R. E. Crochiere, "A novel implementation for narrow-band FIR digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 457–464, Oct. 1975.

[13] Z. Jing and A. T. Fam, "A new structure for narrow transition band, lowpass digital filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 362–370, Apr. 1984.

[14] S. Zionts, *Linear and Integer Programming*. Englewood Cliffs, NJ: Prentice-Hall, 1974.

[15] L. R. Rabiner, "Linear program design of finite impulse response (FIR) digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 280–288, Oct. 1972.

[16] Y. C. Lim, "Efficient special purpose linear programming for FIR filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 963–968, Aug. 1983.

[17] Y. C. Lim and S. R. Parker, "Discrete coefficient FIR digital filter design based upon an LMS criteria," *IEEE Trans. Circuits Syst.*, vol. CAS-30, pp. 723–739, Oct. 1983.

[18] Y. C. Lim and S. R. Parker, "FIR filter design over a discrete powers-of-two coefficient space," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. ASSP-31, pp. 583–591, June 1983.

[19] Y. C. Lim, S. R. Parker, and A. G. Constantinides, "Finite wordlength FIR filter design using integer programming over a discrete coefficient space," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 661–664, Aug. 1982.

[20] Y. C. Lim, "Predictive coding for FIR filter wordlength reduction," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 365–372, Apr. 1985.

[21] D. M. Kodek and K. Steiglitz, "Comparison of optimal and local search methods for designing finite wordlength FIR digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-28, pp. 28–32, Jan. 1981.

[22] D. M. Kodek, "Design of optimal finite wordlength FIR digital filters using integer programming techniques," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 304–308, June 1980.

Yong Ching Lim (S'79–M'82) was born in Malaysia. He received the ACGI and B.Sc degrees in 1977 and the DIC and Ph.D. degrees in 1980, all in electrical engineering from Imperial College, University of London, London, U.K.

From 1980 to 1982, he was a National Research Council Research Associate in the Naval Postgraduate School, Monterey, CA. He joined the Department of Electrical Engineering, National University of Singapore, Singapore, in 1982. His research interests include system modeling, finite wordlength digital filter design, adaptive filtering, parallel processing, active *RC* filters, switched-capacitor filters, and radar signal processing.

Dr. Lim is a member of Eta Kappa Nu.