Substitution of (23) in (21) or (22) gives the corresponding maximum allowed relative deviation:

$$\hat{\epsilon}_r = \frac{\sqrt{k(k+1)} - k}{\sqrt{k(k+1)} + k}. \tag{24}$$

Expression (23) represents a further design constraint. It can be interpreted geometrically, in the parameter space, as the line crossing both the axes at the value $\sqrt{k(k+1)}$ (Fig. 4).

Now we want to investigate if there exist parameter values satisfying constraint (23) and such that:

$$\tau_1\left(\frac{\delta}{|\alpha|}\right) \geq \hat{\epsilon}_r \tag{25}$$

$$\min\left\{\tau_2'\left(\frac{\theta}{|\alpha|}\right), \tau_2''\left(\frac{\theta}{|\alpha|}\right)\right\} \geq \hat{\epsilon}_r. \tag{26}$$

If such values exist then $\hat{\epsilon}_r$ represents the maximum relative deviation compatible with all the design constraints. Substitution of (16) in (25) gives:

$$\frac{\delta}{|\alpha|} \geq \hat{\rho} \triangleq \frac{1+\hat{\epsilon}_r}{1-\hat{\epsilon}_r}. \tag{27}$$

On the other hand, (26) gives:

$$(k-1)\hat{\rho} \leq \frac{\theta}{|\alpha|} \leq \frac{k}{\hat{\rho}} \tag{28}$$

There is only one point, in the parameter space, that satisfies simultaneously (23), (27), and (28), that is:

$$\frac{\delta}{|\alpha|} = \hat{\rho}, \frac{\theta}{|\alpha|} = (k-1)\hat{\rho} \tag{29}$$

as can be easily verified taking into account the identity: $\hat{\rho}k = \sqrt{k(k+1)}$. Point (29) represents the optimal parameter choice (Fig. 4). In the particular case $k = 1$, the optimal choice is $\theta = 0$ and $\delta = \sqrt{2}|\alpha|$. The corresponding required accuracy is ≈ 0.17. Fig. 5 shows the behavior of the maximum relative deviation $\hat{\epsilon}_r$ vs. k. Note that, assuming a precision of 1%, it must be $k \leq \approx 20$.

References

[1] J. Lazzaro, S. Ryckebush, M. A. Mahowald, and C. A. Mead, "Winner-take-all networks of o(N) complexity," in D.S. Touretzky, Ed., *Advances in Neural Information Processing Systems.* San Mateo, CA: Morgan Kaufmann, 1989, pp. 703–711.

[2] R. Perfetti, "Winner-take-all circuit for neurocomputing applications," *IEE Proc.*, Part G, vol. 137, no. 5, pp. 353–359, 1990.

[3] E. Majani, R. Erlanson, and Y. Abu-Mostafa, "On the k-winners-take-all network," in D. S. Touretzky, Ed., *Advances in Neural Information Processing Systems.* San Mateo, CA: Morgan Kaufmann, 1989, pp. 634–642.

[4] W. J. Wolfe, D. Mathis, C. Anderson, J. Rothman, M. Gottler, G. Brady, R. Walker, G. Duane, and G. Alaghband, "K-winner networks," *IEEE Trans. Neural Networks*, vol. 2, no. 2, pp. 310–315, 1991.

[5] R. P. Lippman, "Introduction to computing with neural nets," *IEEE ASSP Magazine*, pp. 4–22, Apr. 1987.

[6] G. Seiler and J. A. Nossek, "Winner-take-all cellular neural networks," *IEEE Trans. Circuits Syst.—Part II*, vol. 40, no. 3, pp. 184–190, Mar. 1993.

[7] G. Seiler, A. J. Schuler, and J. A. Nossek, "Design of robust cellular neural networks," *IEEE Trans. Circuits Syst.—Part I*, vol. 40, no. 5, pp. 358–364, May 1993.

[8] J. A. Nossek, G. Seiler, T. Roska, and L. O. Chua, "Cellular neural networks: Theory and circuit design," *Int. J. Circuit Theory and Appl.*, vol. 20, pp. 533–553, 1992.

The Synthesis of Half-Band Filter Using Frequency-Response Masking Technique

T. Saramäki, Y. C. Lim, and R. Yang

***Abstract*—An important property of a half-band filter is that half of its coefficient values are trivial. This yields significant advantage in terms of its computational complexity. Nevertheless, the complexity of a half-band filter is still very high if its transition-width is very narrow. In this letter, we introduce a novel method for the synthesis of very sharp half-band filter using the frequency response masking technique.**

I. Introduction

The structure of a filter synthesized using the frequency response masking technique [1]–[4] is shown in Fig. 1. In this figure, $H_{Ma}(z)$ and $H_{Mc}(z)$ are the masking filters and $H_a(z)$ is obtained by replacing each delay element of a prototype low-pass filter $H_a(z)$ by M delay elements. The transfer function of the filter system of Fig. 1 is given by

$$H(z) = H_{Ma}(z)H_a\left(z^M\right) + \left(z^{-\cdot} - H_a\left(z^M\right)\right)H_{Mc}(z) \tag{1}$$

In order to simplify notation, in this letter, we shall assume that all the filters are zero phase. As a consequence, the resulting filters are noncausal. Nevertheless, causality can be easily achieved by delaying the impulse response of the filter by an appropriate number of samples.

II. Synthesis of Half-Band Filter Using the Frequency Response Masking Technique

Consider a half-band filter of length $4L - 1$ and transfer function $H_a(z)$ given by

$$H_a(z) = \frac{1}{2} + A(z) \tag{2}$$

where

$$A(z) = \sum_{k=1}^{L} a(2k-1)\left[z^{2k-1} + z^{-(2k-1)}\right] \tag{3}$$

In (3), $a(n)$ is the impulse response of the filter at time n and time $-n$. The transfer function of the filter system is then given by

$$H(z) = \left[\frac{1}{2} + A\left(z^M\right)\right]H_{Ma}(z) + \left[\frac{1}{2} - A\left(z^M\right)\right]H_{Mc}(z) \tag{4}$$

If M is odd, then either $\frac{1}{2} + A(z^M)$ or $\frac{1}{2} - A(z^M)$ has a transition band centered at $\frac{1}{2}\pi$ as desired; the sampling frequency is assumed to be 2π. Fig. 2 shows the frequency responses of $H_a(z^M)$, $H_{Ma}(z)$, and $H_{Mc}(z)$ for $M = 4 * integer + 1$. The frequency response plots for $M = 4 * integer + 3$ is similar to that of Fig. 2 with the exception that those for $H_a(z^M)$, $H_{Ma}(z)$, and $H_{Mc}(z)$ are replaced by 1-$H_a(z^M)$, $H_{Mc}(z)$, and $H_{Ma}(z)$, respectively.

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T. Saramäki is with the Signal Processing Laboratory, Tampere University of Technology, SF-33101 Tampere, Finland.

Y. C. Lim and R. Yang are with the Electrical Engineering Department, National University of Singapore, Singapore 0511.

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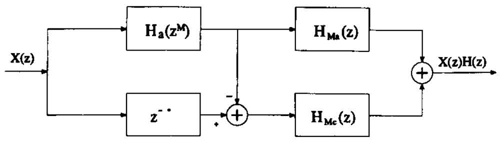


Fig. 1. The structure of a filter synthesized using the frequency-response masking technique.

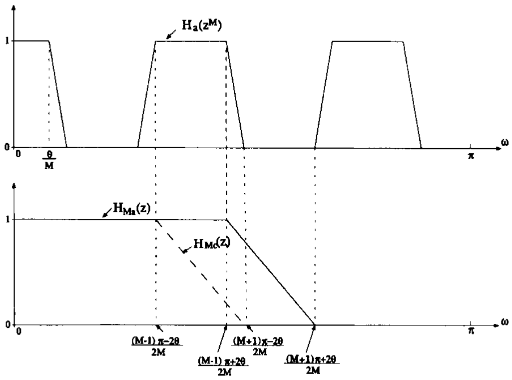


Fig. 2. Frequency responses of $H_a(z^M)$, $H_{Ma}(z)$, and $H_{Mc}(z)$.

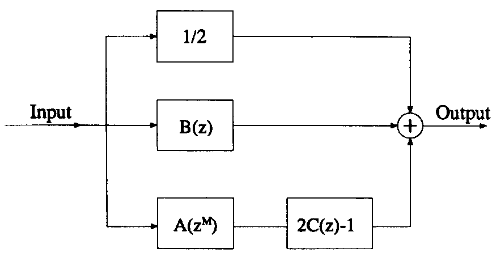


Fig. 3. Synthesis filter structure for halfband filter design.

If the passband and stopband frequency response ripple magnitudes of $H_{Ma}(z)$ and $H_{Mc}(z)$ are equal, then it can be shown that

$$H_{Mc}\left(e^{j\omega}\right) = 1 - H_{Ma}\left(e^{j(\pi-\omega)}\right) \tag{5}$$

where $H_{Ma}(e^{j\omega})$ and $H_{Mc}(e^{j\omega})$ are the frequency responses of $H_{Ma}(z)$ and $H_{Mc}(z)$, respectively. Let the transfer function of $H_{Ma}(z)$ be given by

$$H_{Ma}(z) = h_{Ma}(0) + \sum_{k=1}^{J} h_{Ma}(k)\left(z^k + z^{-k}\right) \tag{6}$$

The length of $H_{Ma}(z)$ is $2J-1$. Let

$$B(z) = h_{Ma}(1)\left(z + z^{-1}\right) + h_{Ma}(3)\left(z^3 + z^{-3}\right) + \ldots \tag{7}$$

Let

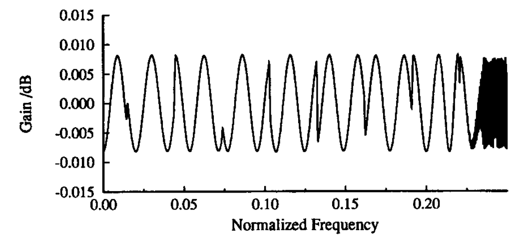
$$C(z) = h_{Ma}(0) + h_{Ma}(2)\left(z^2 + z^{-2}\right) + h_{Ma}(4)\left(z^4 + z^{-4}\right) + \ldots \tag{8}$$

Hence, $H_{Ma}(z)$ can be written as

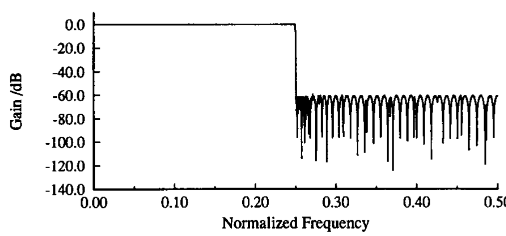
$$H_{Ma}(z) = B(z) + C(z) \tag{9}$$

TABLE I
THE ESTIMATED LENGTHS OF $H_a(z)$ AND $H_{Ma}(z)$, FOR VARIOUS M. N_a AND N_{Ma} ARE THE LENGTHS OF $H_a(z)$ AND $H_{Ma}(z)$, RESPECTIVELY. *Req.mltpls* IS THE REQUIRED NUMBER OF MULTIPLIERS FOR THE SYNTHESIS FILTER OF OUR METHOD.

M	N_a	N_{Ma}	*Req.mltpls*
3	1117	19	289
5	671	33	184
7	479	47	143
9	373	61	123
11	305	75	114
13	259	87	108
15	225	101	107
17	197	113	106
19	177	129	109
21	161	141	111
23	147	155	114
25	135	169	118
27	125	181	122



(a)



(b)

Fig. 4. Frequency response of synthesis filter. (a) Passband response. (b) Overall response.

$H_{Mc}(z)$ is then given by

$$H_{Mc}(z) = 1 - H_{Ma}(-z) = 1 + B(z) - C(z) \tag{10}$$

Substituting (9) and (10) into (4), we have

$$H(z) = \frac{1}{2} + B(z) + A\left(z^M\right)(2C(z) - 1) \tag{11}$$

The synthesis structure for (11) is shown in Fig. 3. Note that $C(z)$ consists of even power terms of z. Apart from the constant term, $A(z^M)$ is a power series consisting of odd power terms of z. As a consequence, the $A(z^M)(2C(z)-1)$ term of (11) consists of a constant term and terms with odd powers of z. Since $B(z)$ is also a power series consisting of odd power terms of z, $H(z)$ consists of a constant term and terms with odd powers of z. Henze, $H(z)$ is a half-band filter.

III. An Example

We shall choose the design of a half-band filter with a transition width of 0.001 sampling frequency and 0.001 peak frequency response ripple magnitude as an example to illustrate our method. It is estimated [5] that this set of specifications can be met by a minimax optimum filter of length 3255; 815 multipliers are required. To meet this set of specifications using our method, the estimated length of $H_a(z)$ and that of $H_{Ma}(z)$ for M ranging from 3 to 27 are tabulated in Table I. As can be seen from Table I, the design with the minimum number of multipliers is the one with $M = 17$; 106 multipliers are required. Its frequency response is shown in Fig. 4.

IV. Conclusion

A method for the synthesis of very sharp half-band filters using the frequency response masking technique is introduced. Our method produces significant savings in the number of multipliers required for implementation.

References

[1] Y. C. Lim, "Frequency-response masking approach for the synthesis of sharp linear phase digital filter," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 357–364, April 1986.

[2] Y. C. Lim and Y. Lian, "The optimum design of one- and two-dimensional FIR filters using the frequency response masking technique," *IEEE Trans. Circuits Syst. II.*, vol. 40, pp. 88–95, Feb. 1993.

[3] T. Saramäki and A. T. Fam, "Subfilter approach for designing efficient FIR filters," *Proc. 1988 Int. Symp. Circuits Syst.*, pp. 2903–2915.

[4] G. Rajan, Y. Neauo, and S. K. Mitra, "On the design of sharp cutoff wide-band FIR filters with reduced arithmetic complexity," *IEEE Trans. Circuits Systs.*, vol. 35, pp. 1447–1454, Nov. 1988.

[5] O. Herrmann, L. R. Rabiner, and D. S. K. Chan, "Practical design rules for optimum finite impulse response lowpass digital filters," *Bell Syst. Tech. J.*, vol. 52, pp. 769–799, July–Aug. 1973.