Efficient Design of Pulse Shaping Filters for OFDM Systems

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ABSTRACT

Orthogonal frequency division multiplexing (OFDM) has recently become a popular technique for high-data-rate transmission over wireless channels. Due to the time-frequency dispersion caused by the channel, the performance of OFDM systems depends critically on the time-frequency localization of the pulse shaping filters. In this paper, we show how the recent *duality and biorthogonality theory*[1,2] developed in the context of Weyl-Heisenberg frames can be used to devise simple and efficient design procedures for well-localized OFDM pulse shaping filters. We consider OFDM systems employing time-frequency guard regions and OFDM systems based on offset QAM. We propose FFT-based design methods for pulse shaping filters with arbitrary length and arbitrary overlapping factors. Finally, we present some design examples.

Keywords: OFDM, pulse shaping filter, Weyl-Heisenberg frames

1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM)[3–10] is employed in current telecommunications standards such as satellite and terrestrial digital audio broadcasting (DAB), digital terrestrial TV broadcasting (DVB),[11] asymmetric digital subscriber line (ADSL) for high-bit-rate digital subscriber services on twisted-pair channels, and broadband indoor wireless systems.[12]

Wireless channels in general introduce both time-and frequency-dispersion. The time-dispersion arises from multipath propagation and the frequency-dispersion is due to Doppler spread, which is caused by moving vehicles. The error resulting from channel dispersion depends critically on the time-frequency localization of the transmitter pulse shaping filter. Good time-frequency localization avoids the symbol energy spreading out and perturbing neighboring symbols in the time-frequency plane.[9,13] The design of well-localized OFDM pulse shaping filters is therefore an important topic.[3,14,15] Besides dispersion-robustness further advantages of pulse shaping are reduced out-of-band emission in wireless OFDM systems[16,17] and reduced sensitivity to synchronization errors.[18–20]

OFDM systems based on offset QAM (OFDM/OQAM)[21] have recently been shown to bypass a major disadvantage of OFDM schemes based on ordinary QAM, namely the fact that time-frequency well-localized (and hence dispersion-robust) pulse shaping filters are prohibited in the case of critical time-frequency lattice density where spectral efficiency is maximized. In signal processing this phenomenon is known as Balian-Low theorem.[22] Pulse shaping OFDM/OQAM systems are therefore well-suited for wireless high-data-rate applications.[15,14] However, since they do not employ time-frequency guard regions such as a cyclic prefix (CP) for example*, equalization is in general more complicated.

In this paper, we provide a unifying framework for the design of pulse shaping OFDM and OFDM/OQAM systems, and we show how recent findings in the theory of Weyl-Heisenberg frames[1,2] can be applied to develop procedures for the design of OFDM pulse shaping filters.

The organization of the paper is as follows. Section 2 reviews pulse shaping OFDM and OFDM/OQAM systems and discusses a fundamental tradeoff between spectral efficiency and pulse shaping filter time-frequency localization. Section 3 presents general orthogonality and biorthogonality conditions. In Section 4, we establish an interesting link between recent results in Weyl-Heisenberg frame theory and OFDM systems, and we show how this connection can be used for designing OFDM pulse shaping filters. Finally, Section 5 provides design examples.

2. PULSE SHAPING OFDM SYSTEMS

In this section, we shall briefly review the different types of pulse shaping OFDM systems considered in the paper.

*Note that CP OFDM systems employ a temporal guard region only.

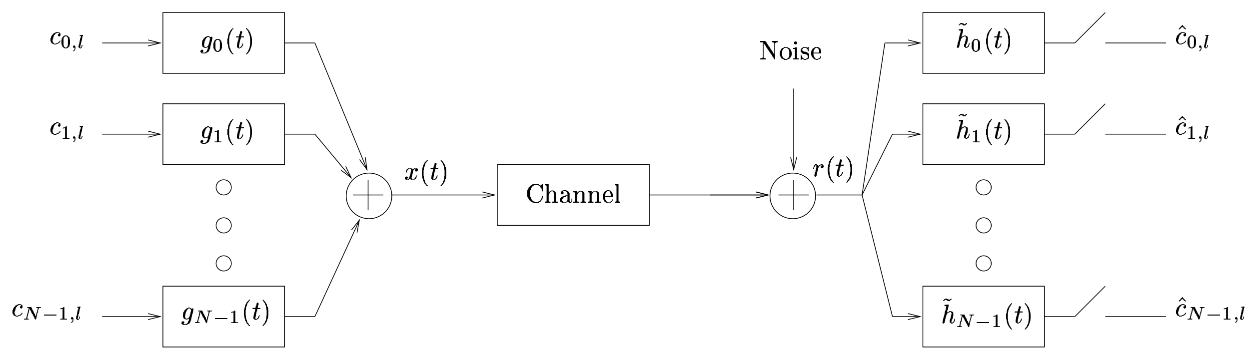

Figure 1. Baseband OFDM system model ($\tilde{h}_l(t) = h_l^*(-t)$).

2.1. OFDM/QAM systems

The baseband equivalent of a pulse shaping OFDM signal is given by

$$x(t) = \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} c_{k,l}\, g(t - lT)\, e^{j2\pi kF(t-lT)}, \tag{1}$$

where T is the OFDM symbol duration, F denotes the subcarrier spacing, N is the number of subcarriers, $g(t)$ is the transmitter pulse shaping filter, and $c_{k,l}$ denotes the data symbols. The reconstructed data symbols are given by $\hat{c}_{k,l} = \langle r, h_{k,l} \rangle$, where $r(t)$ is the received signal and $h_{k,l}(t) = h(t - lT)\, e^{j2\pi kF(t-lT)}$ with $h(t)$ denoting the receiver pulse shaping filter. Fig. 1 shows a baseband OFDM system model. The case $h(t) = g(t)$ will be referred to as OFDM, whereas $h(t) \neq g(t)$ (i.e. different transmitter and receiver pulse shaping filters) will be denoted as biorthogonal frequency division multiplexing (BFDM). Relaxing the orthogonality requirement leads to improved pulse shaping filter quality. For $TF > 1$, the system is said to employ a time-frequency guard region.[13,23] In fact, CP OFDM systems can be seen as a special case thereof with the time-frequency guard region being a temporal guard region only. Note, however, that $TF > 1$ can not only be achieved by insertion of a CP; for example, one can introduce spectral guard regions by spacing the subcarriers further apart in order to avoid ICI in frequency-dispersive environments.

Spectral efficiency. It has been shown in[9,13] that for a great number of subcarriers N, the spectral efficiency of an OFDM system can be approximated by

$$\eta = \frac{\beta}{TF} \qquad \left[\frac{\text{bit/s}}{\text{Hz}}\right],$$

where β is the number of bits per symbol. In other words, the density of the lattice of transmission points in the time-frequency plane determines the spectral efficiency, which is maximized if $TF = 1$ (i.e. $\eta_{\max} = \beta$). Note that $TF < 1$ provides an incomplete representation in the sense that the transmit signal $x(t)$ does not convey all information on the data symbols $c_{k,l}$. This case will therefore be excluded in the following. In the case of OFDM systems employing a cyclic prefix (CP)[4] we get $\eta = \beta\left(1 - \frac{T_{cp}}{T}\right)$, where T_{cp} denotes the length of the CP. In practice $\frac{T_{cp}}{T} = \frac{1}{4}$ and consequently $\eta = \frac{3}{4}\beta$, which implies a reduction in spectral efficiency by 25 percent.

Time-frequency localization and wireless channels. The OFDM transmit signal $x(t)$ in (1) is given by a linear combination of the basis functions $g_{k,l}(t) = g(t - lT)e^{j2\pi kF(t-lT)}$ $(k = 0, 1, ..., N-1,\ l \in \mathbb{Z})$. In the OFDM case demodulation is accomplished by projecting the received signal $r(t)$ onto the basis functions $g_{k,l}(t)$, i.e., $\hat{c}_{k,l} = \langle r, g_{k,l} \rangle$. (This corresponds to matched filtering and symbol rate sampling in the receiver). In the BFDM case, $\hat{c}_{k,l} = \langle r, h_{k,l} \rangle$. If the channel is perfect (no dispersion and no channel noise) orthogonality or biorthogonality between the transmitter and receiver basis functions guarantees intersymbol interference (ISI)-free and intercarrier interference (ICI)-free transmission, i.e., $\hat{c}_{k,l} = c_{k,l}$. However, if the channel is time-frequency dispersive such as the mobile radio channel for example,[24] good time-frequency localization of the basis functions is necessary in order to minimize the distortion due to channel dispersion. In general, there will be both ISI and ICI due to the lack of orthogonality or biorthogonality between the perturbed transmitter basis functions and the receiver basis functions.[9] The amount of the resulting interference depends on the time-frequency localization of the transmitter basis functions. Ideally, one would like the pulse shaping filters $g(t)$ and $h(t)$ to be such that no ISI and no ICI are caused. In this case

$$\hat{c}_{k,l} = B(lT, kF)\, c_{k,l}, \tag{2}$$

where $B(t, f)$ denotes the time-varying transfer function of the channel.[25] Consequently, equalization would reduce to simple divisions in the receiver.

OFDM systems based on a CP (OFDM without pulse shaping) employ a rectangular $g(t)$ which has poor frequency localization and can therefore lead to ICI in frequency-dispersive (rapidly time-varying) channels. Ideal bandpass filters on the other hand have poor temporal localization and can therefore lead to ISI in the case of time-dispersive channels. Thus, in the presence of both time- and frequency-dispersion the optimum solution will employ a pulse shaping filter which is well-localized simultaneously in time and frequency. The optimum design of pulse shaping filters for given channel specifications has been discussed for the case of OFDM/QAM and BFDM/QAM systems employing time-frequency guard regions in.[23] We finally note that the computational complexity of pulse shaping OFDM systems is slightly higher than that of CP OFDM systems.

Spectral efficiency versus time-frequency localization. It is well known from the theory of Weyl-Heisenberg frames[22,26] that time-frequency well-localized (orthogonal or biorthogonal) basis functions of the form $g_{k,l}(t) = g(t-lT)\, e^{j2\pi kF(t-lT)}$ exist only for $TF > 1$. More specifically, it can be shown that a $g(t)$ generating an orthogonal or biorthogonal basis $\{g_{k,l}(t)\}$ for $L^2(\mathbb{R})$ with $TF = 1$ cannot satisfy $\int t^2 |g(t)|^2\, dt < \infty$ and $\int f^2 |G(f)|^2\, df < \infty$ simultaneously, where $G(f) = \int g(t)\, e^{-j2\pi ft}\, dt$ denotes the Fourier transform of $g(t)$. In practice, one needs at least $TF = \frac{3}{2}$ to obtain pulse shaping filters with reasonable time-frequency localization. A recent proposal of a well-localized OFDM/QAM pulse shaping filter[9,13] uses a time-frequency lattice density of $TF = 2$, which implies a reduction in spectral efficiency by 50 percent, i.e., $\eta = \frac{\beta}{2}$. The above analysis unveils a major drawback of OFDM/QAM systems, namely the fact that time-frequency well-localized pulse shaping filters are prohibited in the case of critical time-frequency lattice density $TF = 1$, i.e., maximum spectral efficiency $\eta_{\max} = \beta$. Therefore, when using OFDM/QAM systems a compromise between pulse shaping filter localization (and hence dispersion robustness) and spectral efficiency has to be sought.

2.2. OFDM systems based on offset QAM

It has been pointed out in[9,10,14,15] that in contrast to OFDM/QAM systems† OFDM systems based on OQAM allow well-localized basis functions (with finite time- and frequency-dispersion) even for critical lattice density $TF = 1$, i.e., maximum spectral efficiency. In fact, in this paper, we show that an OFDM/OQAM system with $TF = 1$ achieves the same pulse shaping filter quality (in terms of time-frequency localization) as an OFDM/QAM system with $TF = 2$.

In the following, we consider an N-channel OFDM(BFDM)/OQAM system with $TF = 1$. The corresponding transmit signal is given by

$$x(t) = \sum_{k=0}^{N-1} x_k(t) = \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k,l}^{\mathcal{R}}\, g(t-lT)\, e^{j\frac{2\pi}{T}k(t-\frac{\alpha T}{2N})} + \sum_{l=-\infty}^{\infty} j c_{k,l}^{\mathcal{I}}\, g(t+T/2-lT)\, e^{j\frac{2\pi}{T}k(t-\frac{\alpha T}{2N})} \right], \tag{3}$$

where $c_{k,l}^{\mathcal{R}} = \mathrm{Re}\{c_{k,l}\}$ and $c_{k,l}^{\mathcal{I}} = \mathrm{Im}\{c_{k,l}\}$ denote the real and imaginary parts of the data symbols $c_{k,l}$, respectively, $g(t)$ is the transmitter pulse shaping filter, and $\alpha \in [0, N-1]$. The receiver performs a demodulation according to

$$\begin{aligned} \hat{c}_{k,l}^{\mathcal{R}} &= \int_{-\infty}^{\infty} \mathrm{Re}\left\{ x(\tau)\, e^{-j\frac{2\pi}{T}k(\tau-\frac{\alpha T}{2N})} \right\} g(\tau - lT)\, d\tau \\ \hat{c}_{k,l}^{\mathcal{I}} &= \int_{-\infty}^{\infty} \mathrm{Im}\left\{ x(\tau)\, e^{-j\frac{2\pi}{T}k(\tau-\frac{\alpha T}{2N})} \right\} g\left(\tau + \frac{T}{2} - lT\right) d\tau. \end{aligned}$$

†Note that the symbol constellation in the OFDM systems discussed in Sec. 2.1 is not really limited to QAM. In fact, the $c_{k,l}$ can be taken from an arbitrary constellation. We chose the terminology of OFDM/QAM to emphasize the difference between the OFDM systems discussed in Sec. 2.1 and OFDM based on OQAM discussed in this subsection.

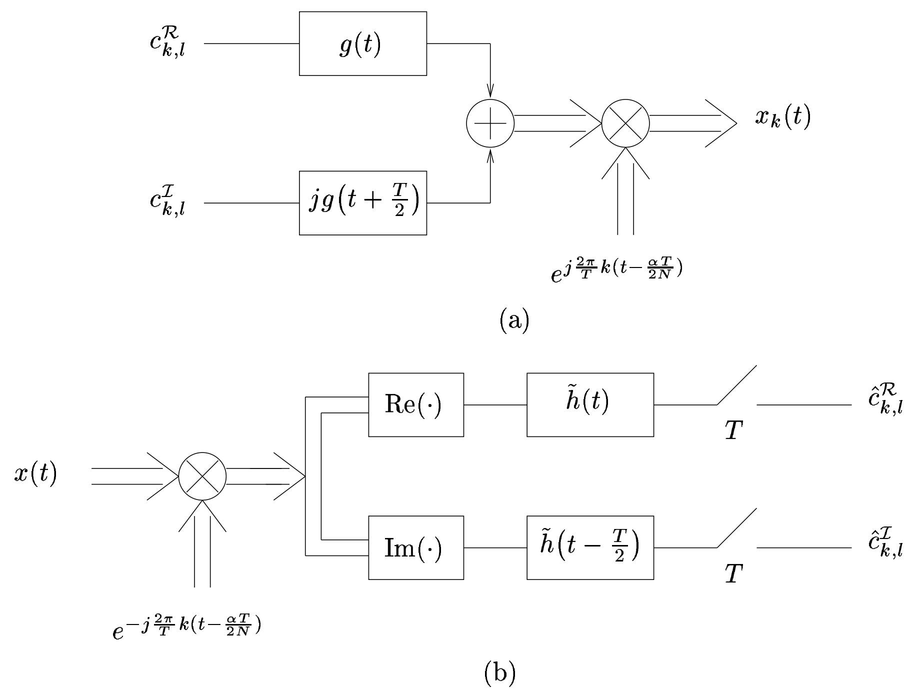

Figure 2. OFDM (BFDM) system based on OQAM: (a) k-th transmitter subchannel, (b) k-th receiver subchannel.

Fig. 2 depicts the k-th transmitter and receiver subchannel of an OFDM(BFDM)/OQAM system, respectively.

3. ORTHOGONALITY AND BIORTHOGONALITY CONDITIONS

OFDM/QAM systems. Assuming that the total bandwidth of the OFDM signal $x(t)$ in (1) is approximately NF and setting‡ $TF = \frac{M}{N} \geq 1$, a critically sampled version of $x(t)$ is obtained as

$$x[n] = \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} c_{k,l}\, g[n-lM] e^{j\frac{2\pi}{N}k(n-lM)}, \tag{4}$$

where $x[n] \triangleq x\left(\frac{n}{NF}\right)$ and $g[n] \triangleq g\left(\frac{n}{NF}\right)$. The reconstructed data symbols are given by $\hat{c}_{k,l} = \langle x, h_{k,l}\rangle$ with $h_{k,l}[n] = h[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}$. Consequently, the filters $g[n]$ and $h[n]$ are biorthogonal, i.e., $\hat{c}_{k,l} = c_{k,l}$ in the absence of a channel if the function sets $\{g[n-lM]\, e^{j\frac{2\pi}{N}k(n-lM)}\}$ and $\{h[n-lM]\, e^{j\frac{2\pi}{N}k(n-lM)}\}$ are biorthogonal, i.e.,

$$\langle g_{k,l}, h_{k',l'}\rangle = \delta[k-k']\delta[l-l'].$$

Straightforward manipulations reveal that the biorthogonality of $g[n]$ and $h[n]$ can equivalently be expressed in terms of their cross-ambiguity function[27] $A^{(g,h)}[k,\theta) = \sum_{n=-\infty}^{\infty} g[n]h[n-k]e^{-j2\pi n\theta}$ as

$$A^{(g,h)}\left[lM, \frac{k}{N}\right) = \delta[l]\delta[k], \quad l \in \mathbb{Z}, k \in [0, N-1]. \tag{5}$$

For $h[n] = g[n]$ Eq. (5) reduces to $A^{(g,g)}\left[lM, \frac{k}{N}\right) = \delta[l]\delta[k]$ with the auto-ambiguity function $A^{(g,g)}[k,\theta)$ of $g[n]$.[27]

OFDM/OQAM systems. In the OFDM/OQAM case we have $TF = \frac{M}{N} = 1$, which assuming that M is even yields

$$x[n] = \sum_{k=0}^{M-1} \left[\sum_{l=-\infty}^{\infty} c^{\mathcal{R}}_{k,l}\, g[n-lM]\, e^{j\frac{2\pi}{M}k(n-\alpha/2)} + \sum_{l=-\infty}^{\infty} jc^{\mathcal{I}}_{k,l}\, g[n+M/2-lM]\, e^{j\frac{2\pi}{M}k(n-\alpha/2)} \right]. \tag{6}$$

‡Note that usually N is very large, so that this choice for TF does not impose a severe restriction on the possible values of TF.

In[14] it has been shown that a $g[n]$ satisfying the symmetry condition

$$g[n] = g[\alpha + (2r+1)M/2 - n] \tag{7}$$

is orthogonal if the function set $\left\{ g[n-lM] e^{j\frac{2\pi}{M/2}k(n-lM)} \right\}$ is orthogonal, or equivalently

$$A^{(g,g)}\left[lM, 2\frac{k}{M}\right) = \delta[l]\,\delta[k], \qquad l \in \mathbb{Z},\, k \in [0, M/2-1].$$

Here, $\alpha = (L_g + M/2 - 1) \bmod M$, $r \in \mathbb{Z}$, and L_g denotes the length of $g[n]$. Biorthogonality conditions for the OQAM case are slightly more involved and can be found in.[14]

4. OFDM SYSTEMS AND WEYL-HEISENBERG FRAMES

In this section, we shall show how the recently developed *duality and biorthogonality theory* for Weyl-Heisenberg frames[1,2] can be applied to the design of OFDM pulse shaping filters. Let us start with a brief review of Weyl-Heisenberg frames.

4.1. Weyl-Heisenberg frames

For $K \geq L$, a set of functions $g_{k,l}[n] = g[n-lL]\, e^{j\frac{2\pi}{K}k(n-lL)}$ with $-\infty < l < \infty$ and $0 \leq k \leq K-1$ is said to be a Weyl-Heisenberg frame (WHF) for $l^2(\mathbb{Z})$ if for all $x[n] \in l^2(\mathbb{Z})$

$$A\,\|x\|^2 \;\leq\; \sum_{l=-\infty}^{\infty} \sum_{k=0}^{K-1} \left|\langle x, g_{k,l}\rangle\right|^2 \leq\; B\,\|x\|^2 \qquad \text{with} \quad 0 < A \leq B < \infty. \tag{8}$$

The constants $A > 0$ and $B < \infty$ are called *frame bounds*. The cases $K = L$, $K > L$, and $K < L$ will be referred to as critical sampling, oversampling, and undersampling, respectively. For $g[n]$ such that $\{g_{k,l}[n]\}$ is a WHF, the dual frame is given by $\{\gamma_{k,l}^{(0)}[n]\}$ with the Wexler-Raz dual[28,1,2]

$$\gamma^{(0)}[n] = (\mathbf{S}^{-1} g)[n]\,. \tag{9}$$

Here, $\mathbf{S}^{-1}$ is the inverse of the *frame operator* $\mathbf{S}$ defined as

$$(\mathbf{S}x)[n] = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{K-1} \langle x, g_{k,l}\rangle\, g_{k,l}[n].$$

The frame operator is a linear, positive definite operator mapping $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z})$. If $\{g_{k,l}[n]\}$ is a WHF with frame bounds A and B, then $\{\gamma_{k,l}^{(0)}[n]\}$ is a WHF with frame bounds $\tilde{A} = 1/B$ and $\tilde{B} = 1/A$. A WHF is called *snug* if $A \approx B$ and *tight* if $A = B$. For a tight WHF, $\mathbf{S} = A\mathbf{I}$ where $\mathbf{I}$ is the identity operator on $l^2(\mathbb{Z})$, and hence there is simply $\gamma^{(0)}[n] = \frac{1}{A}\, g[n]$. From a $g[n]$ generating a WHF, we can derive a "tight" window function $g_T[n]$ (i.e., a function generating a tight WHF with frame bound $A = 1$) as

$$g_T[n] = (\mathbf{S}^{-1/2} g)[n]\,, \tag{10}$$

where $\mathbf{S}^{-1/2}$ is the inverse of the positive definite operator square root of $\mathbf{S}$. In the following, we shall also need the discrete Zak transform (DZT),[29,30] which is defined as

$$\mathcal{Z}_g(n,\theta) = \sum_{r=-\infty}^{\infty} g[n+rL]\, e^{-j2\pi r\theta}.$$

Finally, we say that two functions $g[n]$ and $h[n]$ are dual if

$$x[n] = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{K-1} \langle x, h_{k,l}\rangle\, g_{k,l}[n] \qquad \forall \quad x[n] \in l^2(\mathbb{Z}). \tag{11}$$

Note that duality does not necessarily imply that $h[n]$ is the Wexler-Raz dual of $g[n]$. In general, in the oversampled case for a given $g[n]$ generating a WHF for $l^2(\mathbb{Z})$ there is an infinite number of $h[n]$ satisfying (11).[22]

4.2. Duality and biorthogonality

OFDM/QAM systems. In Sec. 3 we have seen that $g[n]$ is an orthogonal pulse shaping filter for an OFDM system with symbol length M and N subchannels if the function set $\{g[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}\}$ is orthogonal. Since $M > N$ this corresponds to designing an undersampled orthogonal WH set. We have furthermore seen that the two filters $g[n]$ and $h[n]$ form a biorthogonal pair of OFDM pulse shaping filters if the two function sets $\{g[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}\}$ and $\{h[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}\}$ are biorthogonal. Using a theory recently developed in the context of WHFs,[1,2] we can relate the design of undersampled WH systems (OFDM pulse shaping filters) to oversampled WHFs, which then allows us to apply well-known design procedures. The theory we are going to use is now commonly referred to as *duality and biorthogonality theory.* The two main results that we are going to need are stated below.

Theorem 1.[1,2] The function sets $\{g[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}\}$ and $\{h[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}\}$ with $M \geq N$ are biorthogonal if and only if the associated oversampled function sets $\{g[n-lN]e^{j\frac{2\pi}{M}k(n-lN)}\}$ and $\{h[n-lN]e^{j\frac{2\pi}{M}k(n-lN)}\}$ are dual, i.e., $g[n]$ and $h[n]$ are dual in the sense of (11) for $L = N$ and $K = M$.

Corollary 2.[1,2] For $g[n]$ generating a tight WHF $\{g[n-lN]\,e^{j\frac{2\pi}{M}k(n-lN)}\}$ for $l^2(\mathbb{Z})$ with $M \geq N$ the associated function set $\{g[n-lM]e^{j\frac{2\pi}{N}k(n-lM)}\}$ is orthogonal.

Note that Corollary 2 is an immediate consequence of Theorem 1.

The design of OFDM pulse shaping filters for symbol length M and N subchannels, where $M \geq N$ has therefore been reduced to the design of WHFs with $K = M$ subchannels and time-shift parameter $L = N$. Note that the WHF is oversampled by a factor of M/N. In particular, the design of an orthogonal pulse shaping filter has been reduced to the design of a tight WHF.

Theorem 1 implies that for $M > N$ for a given transmitter pulse shaping filter the receiver pulse shaping filter is not unique. A convenient choice for $h[n]$ is the Wexler-Raz dual $\gamma^{(0)}[n]$, whose efficient computation will be discussed in Sec. 4.3.

We note that using alternative approaches, it has been realized previously in[31,32] that the design of OFDM/QAM pulse shaping filters for symbol length M and N subchannels is equivalent to the design of tight WHFs with $L = N$ and $K = M$.

OFDM/OQAM systems. In Sec. 3 we showed that $g[n]$ constitutes an OFDM/OQAM pulse shaping filter if it is symmetric, i.e., $g[n] = g[\alpha + (2r+1)\frac{M}{2} - n]$ and the associated function set $\{g[n-lM]\,e^{j\frac{2\pi}{M/2}k(n-lM)}\}$ is orthogonal. Using Corollary 2 it follows that a $g[n]$ constituting a tight WHF for $L = \frac{M}{2}$ and $K = M$ yields an orthogonal set $\{g[n-lM]\,e^{j\frac{2\pi}{M/2}k(n-lM)}\}$. This shows that a symmetric OFDM/QAM prototype for $M/N = 2$ yields orthogonality in an OFDM/OQAM system with $M/N = 1$, which proves that for fixed pulse shaping filter quality (in terms of time-frequency localization) OFDM/OQAM systems have twice the spectral efficiency of OFDM/QAM systems. In[14] it has been shown that a BFDM/OQAM pair $\{g[n], h[n]\}$ is obtained by taking a $g[n] = g[\alpha + (2r+1)M/2 - n]$ generating a WHF for $L = \frac{M}{2}$ and $K = M$ and computing the corresponding $\gamma^{(0)}[n]$. Note that in the OFDM/OQAM case in contrast to the OFDM/QAM case with $M > N$, the receiver pulse shaping filter is uniquely determined.

The design of an OFDM/OQAM pulse shaping filter is therefore equivalent to the design of a tight WHF with oversampling factor 2. Furthermore, the computation of the receiver pulse shaping filter in a BFDM/OQAM system has been reduced to the computation of the Wexler-Raz dual of a $g[n]$ generating a WHF with oversampling factor 2.

4.3. Zibulski-Zeevi representation of the frame operator

Efficient methods for constructing tight frames and computing dual frames can be obtained using the Zibulski-Zeevi representation of the WHF operator.[33] In the following we set $L = N, K = M$, and $\frac{M}{N} = \frac{P}{Q} \geq 1$ with $\gcd(P,Q) = 1$. The action of the WHF operator $\mathbf{S}$ can be expressed in the DZT-domain according to

$$\mathcal{Z}_{\mathbf{S}x}(n,\theta) = \frac{N}{Q}\sum_{q=0}^{Q-1}\mathcal{Z}_x(n-qM,\theta)\sum_{p=0}^{P-1}\mathcal{Z}_g\left(n,\theta-\frac{p}{P}\right)\mathcal{Z}_g^*\left(n-qM,\theta-\frac{p}{P}\right). \tag{12}$$

Defining $y[n] = (\mathbf{S}x)[n]$, and the $Q \times 1$ vectors

$$\begin{aligned}
\mathbf{z}_y(n,\theta) &= [\mathcal{Z}_y(n,\theta)\ \mathcal{Z}_y(n-M,\theta)\ \ldots\ \mathcal{Z}_y(n-(Q-1)M,\theta)]^T \\
\mathbf{z}_x(n,\theta) &= [\mathcal{Z}_x(n,\theta)\ \mathcal{Z}_x(n-M,\theta)\ \ldots\ \mathcal{Z}_x(n-(Q-1)M,\theta)]^T,
\end{aligned}$$

we obtain

$$\mathbf{z}_y(n,\theta) = \mathbf{S}(n,\theta)\,\mathbf{z}_x(n,\theta)$$

with the $Q \times Q$ Zibulski-Zeevi matrix ($k = 0,1,...,Q-1, l = 0,1,...,Q-1$)

$$[\mathbf{S}(n,\theta)]_{k,l} = \frac{N}{Q}\sum_{p=0}^{P-1} \mathcal{Z}_g\left(n - kM, \theta - \frac{p}{P}\right)\mathcal{Z}_g^*\left(n - lM, \theta - \frac{p}{P}\right).$$

We have thus represented the frame operator $\mathbf{S}$ in terms of N matrices $\mathbf{S}(n,\theta)$ ($n = 0,1,...,N-1$) of size $Q \times Q$. Consequently $\mathbf{S}^{-1}$ and $\mathbf{S}^{-1/2}$ are represented by $\mathbf{S}^{-1}(n,\theta)$ and $\mathbf{S}^{-1/2}(n,\theta)$, respectively. The case of integer oversampling ($Q = 1$) deserves special attention. For integer oversampling, (12) simplifies to

$$\mathcal{Z}_{\mathbf{S}x}(n,\theta) = \lambda_g(n,\theta)\,\mathcal{Z}_x(n,\theta) \qquad \text{with} \quad \lambda_g(n,\theta) = N\sum_{p=0}^{P-1}\left|Z_g\left(n, \theta - \frac{p}{P}\right)\right|^2. \tag{13}$$

Hence, the frame operator becomes a simple multiplication operator in the DZT domain. Consequently, the inverse frame operator $\mathbf{S}^{-1}$ can be expressed in the DZT domain as a pointwise division, i.e.,

$$\mathcal{Z}_{\mathbf{S}^{-1}x}(n,\theta) = \frac{\mathcal{Z}_x(n,\theta)}{\lambda_g(n,\theta)}. \tag{14}$$

This implies that $\gamma^{(0)}[n]$ can be calculated via the DZT using

$$\mathcal{Z}_{\gamma^{(0)}}(n,\theta) = \frac{\mathcal{Z}_g(n,\theta)}{\lambda_g(n,\theta)} \tag{15}$$

and deriving $\gamma^{(0)}[n]$ according to $\gamma^{(0)}[n] = \int_0^1 \mathcal{Z}_\gamma^{(0)}(n,\theta)\,d\theta$. From an arbitrary prototype function $g[n]$ generating a WHF, a "tight" window function $g_T[n]$ generating a tight WHF with frame bound $A = 1$ can be derived according to

$$\mathcal{Z}_{g_T}(n,\theta) = \frac{\mathcal{Z}_g(n,\theta)}{\sqrt{\lambda_g(n,\theta)}}. \tag{16}$$

We can therefore see, that the inversion of the frame operator (computation of the receiver pulse shaping filter) requires the inversion of N matrices of size $Q \times Q$. The construction of tight WH frames according to (10) reduces to the factorization of N matrices of size $Q \times Q$. In practice this inversion or factorization has to be done for each $n \in [0, N-1]$ for a finite number of frequencies only. The computation of the dual frame and the construction of tight frames become particularly simple in the case of integer oversampling, where the required matrix inversions and factorizations reduce to divisions and taking the square root of a scalar function, respectively. Note furthermore, that the forward and inverse DZT can be implemented efficiently using the FFT.[29]

4.4. Summary of algorithms

We shall next summarize algorithms for constructing orthogonal pulse shaping filters and for computing receiver pulse shaping filters for given transmitter pulse shaping filter. The design of orthogonal pulse shaping filters (OFDM/QAM and OFDM/OQAM cases) can in principle be accomplished by performing a constrained optimization with the side constraints given by the orthogonality conditions. However, for large filter lengths this approach will be computationally very expensive and might furthermore have convergence problems. In the following, we shall therefore introduce a computationally efficient method based on an orthogonalization procedure. The basic idea is to start from an arbitrary (nonorthogonal) filter $g[n]$, which is modified using $\mathbf{S}^{-1/2}$ to obtain an orthogonal pulse shaping filter.

Construction of OFDM/QAM pulse shaping filter. For an OFDM/QAM system with symbol length M and N subchannels perform the following steps to obtain an orthogonal pulse shaping filter $g_o[n]$.

- Design an initial filter $g[n]$.

- Compute the Zibulski-Zeevi matrices $\mathbf{S}(n,\theta)$ $(n = 0, 1, ..., N-1)$ corresponding to the WHF generated by $g[n]$ for $L = N$ and $K = M$.
- Perform matrix factorizations to obtain the matrices $\mathbf{S}^{-1/2}(n,\theta)$ $(n = 0, 1, ..., N-1)$.
- Compute
$$\mathbf{z}_{g_o}(n,\theta) = \mathbf{S}^{-1/2}(n,\theta)\mathbf{z}_g(n,\theta)$$
and perform an inverse DZT to obtain the orthogonal pulse shaping filter $g_o[n]$.

Computing the receiver pulse shaping filter in the OFDM/QAM case. For an OFDM/QAM system with transmitter pulse shaping filter $g[n]$, symbol length M and N subchannels perform the following steps to obtain the receiver pulse shaping filter $h[n]$.

- Compute the Zibulski-Zeevi matrices $\mathbf{S}(n,\theta)$ $(n = 0, 1, ..., N-1)$ corresponding to the WHF generated by $g[n]$ for $L = N$ and $K = M$.
- Perform matrix inversions to obtain the matrices $\mathbf{S}^{-1}(n,\theta)$ $(n = 0, 1, ..., N-1)$.
- Compute
$$\mathbf{z}_h(n,\theta) = \mathbf{S}^{-1}(n,\theta)\mathbf{z}_g(n,\theta)$$
and perform an inverse DZT to obtain the receiver pulse shaping filter $h[n]$.

Construction of OFDM/OQAM pulse shaping filter. For an OFDM/OQAM system with even symbol length M perform the following steps to obtain an orthogonal pulse shaping filter $g_o[n]$.

- Design an initial filter $g[n]$ satisfying the symmetry condition (7).
- Set $L = M/2$ and compute the DZT of the orthogonal filter $g_o[n]$ according to
$$\mathcal{Z}_{g_o}(n,\theta) = \frac{2\mathcal{Z}_g(n,\theta)}{\sqrt{M|\mathcal{Z}_g(n,\theta)|^2 + M\left|\mathcal{Z}_g(n,\theta-\frac{1}{2})\right|^2}}.$$
- Perform an inverse DZT to obtain the orthogonal pulse shaping filter $g_o[n]$.

In[14] it is shown that the resulting pulse shaping filter $g_o[n]$ satisfies the symmetry property (7) if the initial filter $g[n]$ does.

Computing the receiver pulse shaping filter in the OFDM/OQAM case. For an OFDM/OQAM system with transmitter pulse shaping filter $g[n]$ satisfying (7) and even symbol length M perform the following steps to obtain the receiver pulse shaping filter $h[n]$.

- Set $L = M/2$ and compute the DZT of the receiver pulse shaping filter $h[n]$ according to
$$\mathcal{Z}_h(n,\theta) = \frac{4\mathcal{Z}_g(n,\theta)}{M|\mathcal{Z}_g(n,\theta)|^2 + M\left|\mathcal{Z}_g(n,\theta-\frac{1}{2})\right|^2}.$$
- Perform an inverse DZT to obtain the receiver pulse shaping filter $h[n]$.

Note that in the BFDM/OQAM case, we are free to choose any symmetric transmitter pulse shaping filter as long as
$$|\mathcal{Z}_g(n,\theta)|^2 + \left|\mathcal{Z}_g\left(n,\theta-\frac{1}{2}\right)\right|^2 \neq 0, \quad n \in \left[0, \frac{M}{2}-1\right], \theta \in [0,1)$$
is guaranteed.

We finally note that the orthogonalization procedures described above do not automatically guarantee that $g_o[n]$ is well-localized in time and frequency. In practice, however, as we shall see later (Section 5), starting from a well-localized initial filter (i.e. a lowpass filter or a gaussian function with bandwidth approximately equal to $\frac{1}{2N}$ in the OFDM/QAM case and $\frac{1}{M}$ in the OFDM/OQAM case) yields well-localized orthogonal filters.

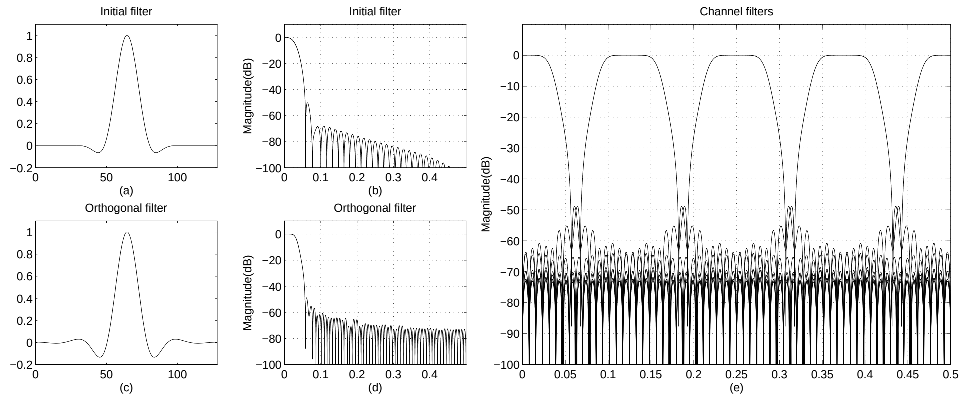

Figure 3. 8-channel OFDM/QAM pulse shaping filter of length 128 with $M = 16$: (a)-(b) initial filter, (c)-(d) orthogonal filter, (e) corresponding channel filters.

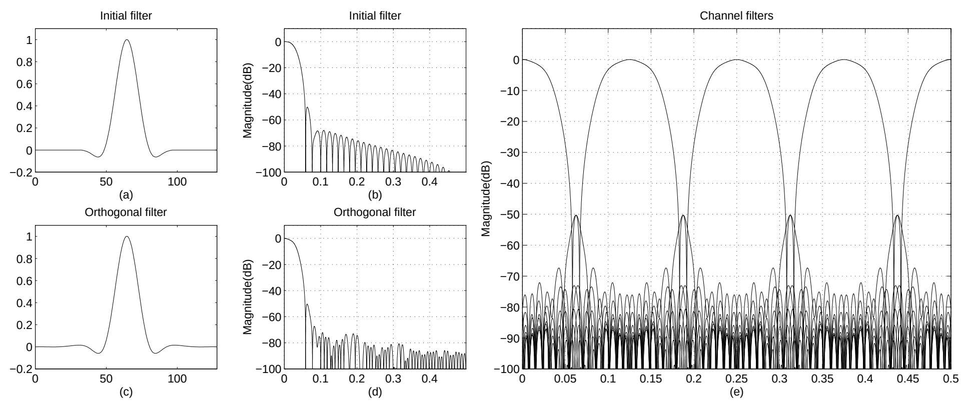

Figure 4. 8-channel OFDM/QAM pulse shaping filter of length 128 with $M = 32$: (a)-(b) initial filter, (c)-(d) orthogonal filter, (e) corresponding channel filters.

5. DESIGN EXAMPLES

We shall finally present some design examples, which have been obtained by applying the methods described in the previous section.

Design example 1. In the first example we used the DZT-based orthogonalization procedure to design an 8-channel OFDM/QAM pulse shaping filter for $M = 16$ and $M = 32$, respectively. In both cases the initial filter was a 64 tap lowpass filter with nominal bandwidth 1/16 (designed using the MATLAB function FIR1) with 32 zeros appended at each end to obtain a prototype of overall length 128. For $M = 16$, Figs. 3(a) and (b) show the initial prototype and its transfer function, and Figs. 3(c) and (d) show the resulting orthogonal prototype and its transfer function. The corresponding channel filters for $M = 16$ are depicted in Fig. 3(e). The initial filter and the orthogonal filter for $M = 32$ are shown in Fig. 4. We can see that in both cases the orthogonal pulse shaping filter has good time-frequency localization. Furthermore, for $M = 32$ this localization is much better than for $M = 16$, which reflects the fact that increasing the amount of time-frequency guard regions yields better filter quality.

Design example 2. In the second design example we used the DZT-based approach to compute the receiver pulse shaping filter for a 32-channel OFDM system with $M = 48$. In this example the initial prototype is a 64 tap lowpass filter with nominal bandwidth 1/32 (designed using the MATLAB function FIR1) with 32 zeros appended at both ends to obtain a prototype of overall length 128. Figs. 5(a) and (b) show the transmitter pulse shaping filter and its transfer function, and Figs. 5(c) and (d) show the resulting receiver pulse shaping filter and its transfer function.

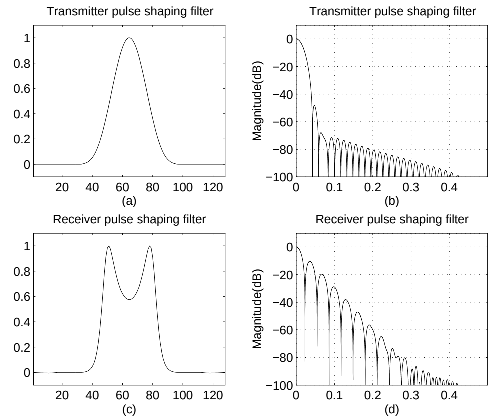

Figure 5. 32-channel OFDM/QAM pulse shaping filter of length 128 ($M = 48$): (a)-(b) transmitter pulse shaping filter, (c)-(d) receiver pulse shaping filter.

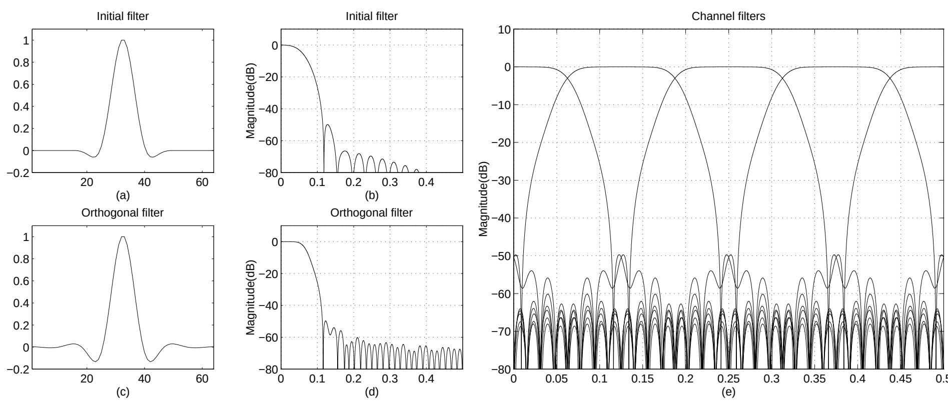

Figure 6. 8-channel OFDM/OQAM pulse shaping filter of length 64: (a)-(b) initial filter, (c)-(d) orthogonal filter, (e) corresponding channel filters.

This example demonstrates that in the biorthogonal case good time-frequency localization of the transmitter pulse shaping filter does not automatically imply the same for the receiver pulse shaping filter. However, unless there is a specific reason for time-frequency well-localized receiver filters this causes no problems.

Design example 3. In this design example we used the DZT-based orthogonalization procedure to design an 8-channel OFDM/OQAM system. Here, the initial prototype is a 32 tap lowpass filter with nominal bandwidth $1/M = 0.125$ (designed using the MATLAB function FIR1) with 16 zeros appended at both ends to obtain an even prototype of overall length 64. Figs. 6(a) and (b) show the initial prototype and its transfer function. Figs. 6(c) and (d) show the resulting orthogonal prototype. The corresponding channel filters are depicted in Fig. 6(e).

Design example 4. In the last example, we demonstrate that our approach allows the design of OFDM systems with a large number of channels and long prototypes. We designed a 256-channel OFDM/OQAM system with prototype length 2048, i.e., the prototype length is 8 times the symbol length. The initial prototype shown in Figs. 8(a) and (b) is a lowpass filter with nominal bandwidth 1/256. Figs. 7(c) and (d) show the resulting orthogonal prototype and its transfer function. Finally, Fig. 7(e) shows the corresponding channel filters.

Acknowledgments

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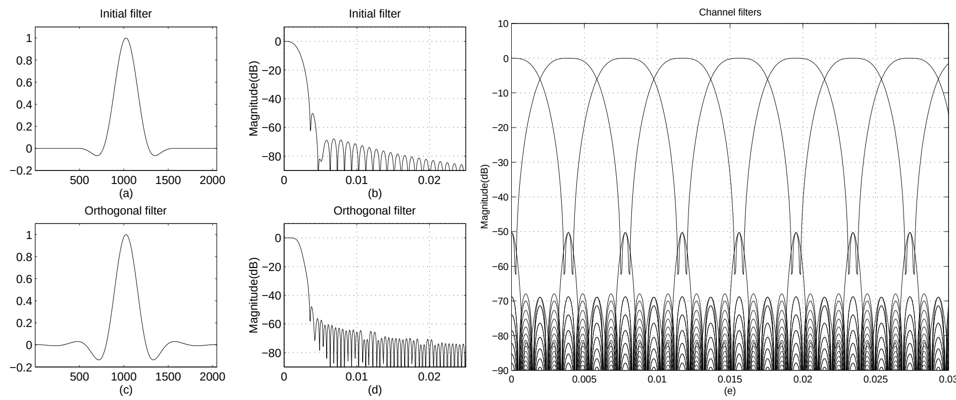

Figure 7. 256-channel OFDM/OQAM pulse shaping filter of length 2048: (a)-(b) initial filter, (c)-(d) orthogonal filter, (e) corresponding channel filters.

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