Generalized Homogeneous Stabilization via Sliding Modes

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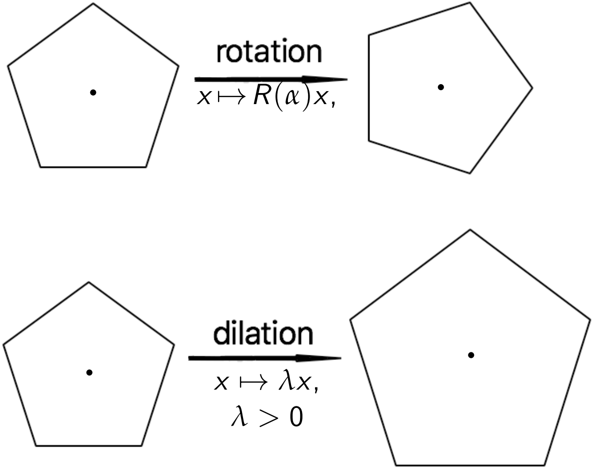
1. Introduction

Homogeneity is a dilation symmetry

Symmetry is an invariance with respect to a group of transformations.



Notation: $\lambda = e^s$, where $s \in \mathbb{R}$ and $e = 2.71828...$ is the Euler number.

Standard and Generalized Homogeneity

- **Linearity** = **Homogeneity** + **Additivity** + **Central Symmetry**

$$f \text{ is linear} \quad \Leftrightarrow \quad f(e^s x) = e^s f(x) \quad \& \quad f(x+y) = f(x) + f(y) \quad \& \quad f(-x) = -f(x)$$

Example: $f(x) = x_1 + x_2$, where $x = (x_1, x_2)^\top$

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- **Standard Homogeneity** [Leonhard Euler, 18th century]:

$x \mapsto e^s x$ (standard dilation) $\qquad f(e^s x) = e^{\nu s} f(x)$, (symmetry)

$s \in \mathbb{R}$ - a scalar parameter $\qquad \nu \in \mathbb{R}$ - homogeneity degree

Example: $f(x) = x_1 x_2 + x_2^2$ is standard homogeneous of degree 2: $f(e^s x) = e^{2s} f(x)$

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Example: $f(x) = x_1 + x_2$, where $x = (x_1, x_2)^\top$ (standard homogeneous of degree 1)

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- **Generalized Homogeneity** [*Zubov 1958; Hermes 1986; Kawski 1991; Rosier 1992;...*]

$x \mapsto \mathbf{d}(s)x$ (a dilation) $\qquad f(\mathbf{d}(s)x) = e^{\nu s} f(x)$ (symmetry)

$\mathbf{d}(s)$ must satisfy certain properties to be a dilation

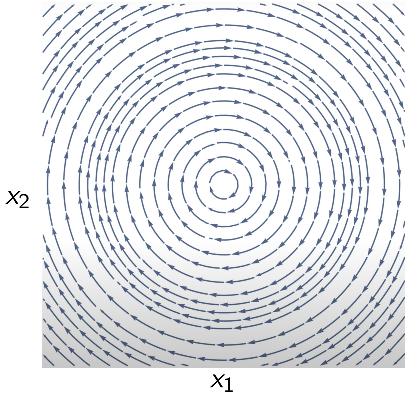
Example: $\mathbf{d}(s) = \begin{pmatrix} e^{2s} & 0 \\ 0 & e^s \end{pmatrix}$, $f(x) = x_1 + x_2^2$ is $\mathbf{d}$-homogeneous: $f(\mathbf{d}(s)x) = e^{2s} f(x)$

Phase portrait of the homogeneous system $\dot{x} = f(x)$

Linear oscillator

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1, \end{cases} \quad x_1, x_2 \in \mathbb{R}$$

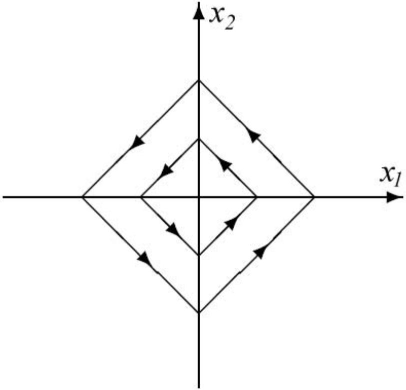
$$f(e^s x) = e^s f(x)$$



Relay oscillator

$$\begin{cases} \dot{x}_1 = -\,\mathrm{sign}[x_2], \\ \dot{x}_2 = \mathrm{sign}[x_1], \end{cases} \quad x_1, x_2 \in \mathbb{R}$$

$$f(e^s x) = e^{0s} f(x)$$

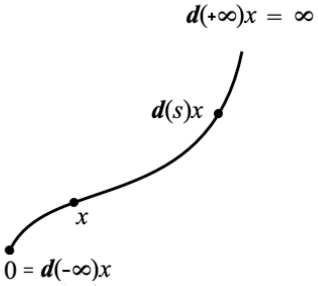


Geometric Homogeneity [Kawski 1991]

Definition

A one-parameter family of operators $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$, $s \in \mathbb{R}$ is said to be a one-parameter group of dilations (or, simply, **dilation**) in $\mathbb{R}^n$ if

- (*Group property*) $\mathbf{d}(0)x = x$ and $\mathbf{d}(s)\circ\mathbf{d}(t)x = \mathbf{d}(t+s)x$;
- (*Limit property*) $\lim\limits_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ and $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$, $x \neq 0$.



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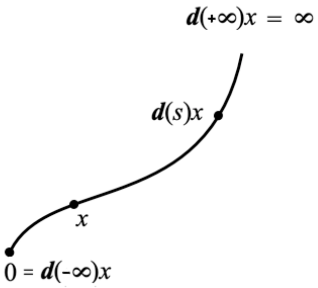

Geometric dilation

$$\mathbf{d}(s)x := \varphi_x(s)$$

is a solution of an unstable ODE

$$\dot{\varphi}(s) = g(\phi(s)), \quad \varphi(0) = x,$$

where $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

Linear (Geometric) Dilation if $\dot{\varphi}(s) = G_{\mathbf{d}}\varphi(s)$

The linear (geometric) **dilation** in $\mathbb{R}^n$ is a matrix-valued function given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \qquad s \in \mathbb{R}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix called a **generator** of $\mathbf{d}$.

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Standard dilation

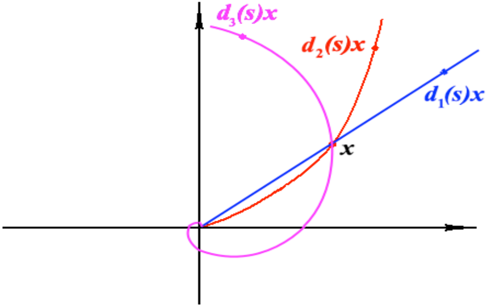
$$\mathbf{d}_1(s) = e^s I_n, \quad G_{\mathbf{d}} = I_n \in \mathbb{R}^{n\times n}$$

Weighted dilation

$\mathbf{d}_2(s) = \mathrm{diag}\{e^{r_i s}\}$, $G_{\mathbf{d}} = \mathrm{diag}\{r_i\} \succ 0$

Linear dilation

$\mathbf{d}_3(s) = e^{sG_d}$, $G_{\mathbf{d}}$ is anti-Hurwitz



Homogeneous functions and vector fields

The definitions are inspired by *Khomenuk 1961 and Kawski 1991*.

Definition (**d**-homogenous function)

A function $h : \mathbb{R}^n \mapsto \mathbb{R}$ is said to **d**-homogeneous of degree $\nu \in \mathbb{R}$ if

$$h(\mathbf{d}(s)x) = e^{\nu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad s \in \mathbb{R}.$$

Definition (**d**-homogenous vector field)

A vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to **d**-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad s \in \mathbb{R}.$$

Example (Minimum time control system)

- The minimum time control problem (Feldbaum 1949, La Salle 1953, Fuller 1960)

$$T \to \min_u$$

$$\text{subject to} \begin{cases} \dot{x}_1 = x_2, & u \in L^\infty((0,T),\mathbb{R}), |u(t)| \leq 1 \\ \dot{x}_2 = u, & x_1(T) = x_2(T) = 0, \end{cases}$$

  has the following feedback solution:

$$u(x) = -\operatorname{sign}(|x_2|x_2 + 2x_1)$$

- The weighted dilation: $\mathbf{d}(s) = \left(\begin{smallmatrix} e^{2s} & 0 \\ 0 & e^s \end{smallmatrix}\right)$
- The dilation symmetry (homogeneity) of the control:

$$u(\mathbf{d}(s)x) = -\operatorname{sign}(|e^s x_2|e^s x_2 + 2e^{2s}x_1) = -e^{0s}\operatorname{sign}(|x_2|x_2 + 2x_1)$$

- The dilation symmetry of the closed-loop system $\dot{x} = f(x) = (x_2 \;\; u(x))^\top$:

$$f(\mathbf{d}(s)x) = \begin{pmatrix} e^s x_2 \\ u(\mathbf{d}(s)x) \end{pmatrix} = \begin{pmatrix} e^s x_2 \\ u(x) \end{pmatrix} = e^{-s}\begin{pmatrix} e^{2s} x_2 \\ e^s u(x) \end{pmatrix} = e^{-s}\mathbf{d}(s)f(x)$$

Historical Remarks

- The types of homogeneity in $\mathbb{R}^n$:

Standard	$\subset$	**Weighted**	$\subset$	**Linear**	$\subset$	**Geometric**
[*Euler 18th cent*]		[*Zubov 1958*]		[*Khomenuk 1961*]		[*Kawski 1991*]

- Homogeneity in Control Systems:

Stability	**Controllability**	**Control Design**	**ISS Analysis**
Zubov 1957	Hermes 1982	Adreini et al 1988	Ryan 1995
Rosier 1992	Kawski 1991	Coron & Praly 1991	Hong 2001
Bhat & Bernstien 1998	Sepulchre & Aeyels 1996	Praly 1997	Andrieu et al 2008
...	...	...	...

	Sliding Mode Control and Estimation	**...**
...	Levant 2003, Orlov 2005,	...
...	Perrruquetti, Floquet & Moulay 2008,	...
...	Dinuzzo & Ferrara 2009, Moreno 2010	...
	...	

2. Homogeneity vs Linearity in Systems and Control

Properties of linear vs homogeneous system

	Linear System $\dot{x} = Ax, \quad x(0) = x_0$ $A \in \mathbb{R}^{n\times n}$ is a matrix	**Homogeneous System** $\dot{x} = f(x), \quad x(0) = x_0$ $f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s)f(x)$

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Trajectory Scaling	$x(t, e^s x_0) = e^s x(t, x_0)$	$x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\mu s}t, x_0)$

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Local $\Leftrightarrow$ Global	$\checkmark$	$\checkmark$

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Local $\Leftrightarrow$ Global	✓	✓
Invariance $\Leftrightarrow$ Stability	✓	✓

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Local $\Leftrightarrow$ Global	✓	✓
Invariance $\Leftrightarrow$ Stability	✓	✓
Stability $\Rightarrow$ Robustness (Input-to-State Stability)	$\dot{x} = Ax + w$ $w \in L^\infty$	$\dot{x} = f(x, w)$ $w \in L^\infty$

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Convergence Rate	Exponential	+Finite/Fixed-time $(\mu \neq 0)$

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Convergence Rate	Exponential	+Finite/Fixed-time $(\mu \neq 0)$
Lyapunov Function	A weighted Euclidean norm $V(x) = \sqrt{x^\top P x}, \quad P \succ 0$	A homogeneous norm $V(\mathbf{d}(s)x) = e^s V(x)$

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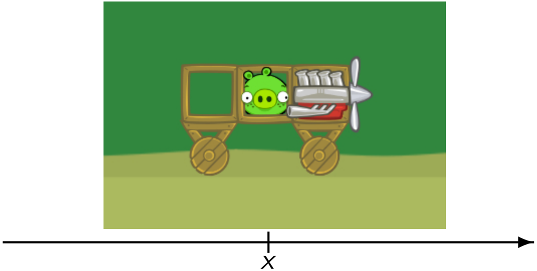
Lyapunov Function	A weighted Euclidean norm $V(x) = \sqrt{x^\top Px}, \quad P \succ 0$	A homogeneous norm $V(\mathbf{d}(s)x) = e^s V(x)$
Consistent discretization (preserves convergence rate)	$\checkmark$ Exponential	$\checkmark$ +Finite/Fixed-time $(\mu \neq 0)$

Question: Is there any potential advantage of a homogeneous controller vs linear one?

Homogeneity vs Linearity: Faster convergence



$$m\dot{v} = -(k_d + k_v v^2)\,\mathrm{sign}(v), \quad t > 0$$

m - mass, $v = \dot{x}$ - velocity, k_d, k_v - coefficients of dry and viscous friction

Fixed-time stability:

$$v(t) = 0 \ \text{ for all } \ t \geq T_{\max} = \frac{m\pi}{2\sqrt{k_d k_v}} \ \text{ and } \ \text{ for any } \ v(0) \in \mathbb{R}^2$$

Homogeneity vs Linearity: Improved robustness

The control aim is to stabilize the scalar system

$$\dot{x} = g + u$$

where g is an uncertainty and u is a control

Model of uncertainty	$\lvert g\rvert \leq c + \lambda\lvert x\rvert$ where $\lambda, c \in \mathbb{R}$ are unknown	a structure of g is unknown but $g(x) \to 0$ as $x \to 0$
Linear control	$u = -kx, \quad k > 0$ *unstable* if $k < \lambda$	$u = -kx, \quad k > 0$ *unstable* if $k < \frac{g(x)}{x}$
Homogeneous control	$u = -k\lvert x\rvert x, \quad k > 0$ *globally practically stable* $\limsup\limits_{t\to+\infty} \lvert x(t)\rvert \leq \frac{\lambda+\sqrt{\lambda^2+4ck}}{2k}$	$u = -k\frac{x}{\lvert x\rvert}, \quad k > 0$ *locally finite-time stable* $\frac{d\lvert x\rvert}{dt} \leq -k + \lvert g(x)\rvert$

Linearity vs Homogeneity: No "peaking" effect (1)

$$\begin{cases} \dot{x} = Ax + bu(x), \\ \|x(0)\| \leq 1, \end{cases} \quad t > 0, \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

where $x = (x_1, x_2, ..., x_n)^\top$, $u : \mathbb{R}^n \to \mathbb{R}$.

The control aim: $\|x(t)\| \leq \varepsilon, \quad \forall t \geq T,$ where $\varepsilon > 0,\ T > 0$ are given

[1] Izmailov 1987, Polyak & Smirnov 2016

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- **Linear control**: For any $\varepsilon > 0$ and $T > 0$ there exists $k = (k_1, k_2, ..., k_n)$ such that

$$u_\ell(x) := kx \quad \Rightarrow \quad \|x(t)\| \leq Ce^{-\sigma t} \leq \varepsilon, \quad \forall t \geq T$$

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Unbounded "peaking": There exists[1] $\gamma > 0$ independent of σ such that

$$\sup_{0 \leq t \leq \sigma^{-1}} \sup_{\|x(0)\|=1} \|x(t)\| \geq \gamma\sigma^{n-1} \to +\infty \text{ as } \varepsilon \to 0$$

[1] Izmailov 1987, Polyak & Smirnov 2016

Linearity vs Homogeneity: No "peaking" effect (2)

Homogeneous control: For any $T > 0$ there exists $\tilde{k} = (\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n)$:

$$u_{hom}(x) := \tilde{k}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \quad \Rightarrow \quad \|x(t)\| = 0, \quad \forall t \geq T.$$

Notice that $|u_{hom}| \leq \|\tilde{k}\|$ and the overshoot is independent of $\varepsilon > 0$.

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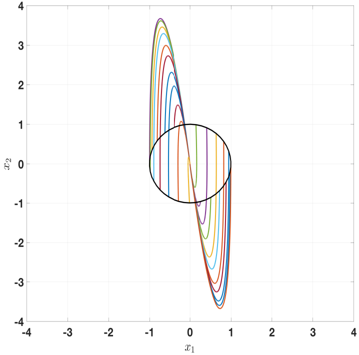
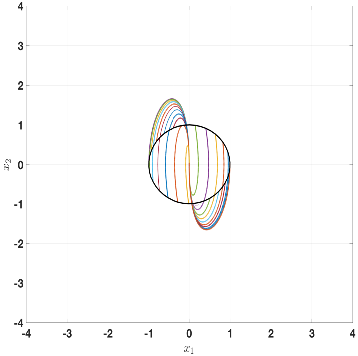
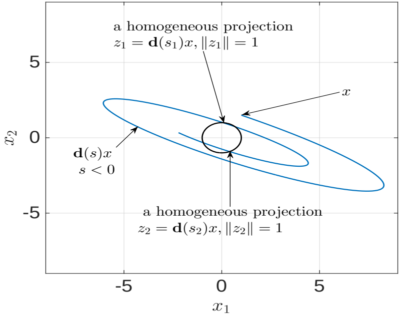




"Overshoots" of linear (left) and homogeneous (right) controllers ($n = 2$, $\varepsilon = 0.005$, $T = 1$)

3. Generalized Homogeneous Euclidean Space

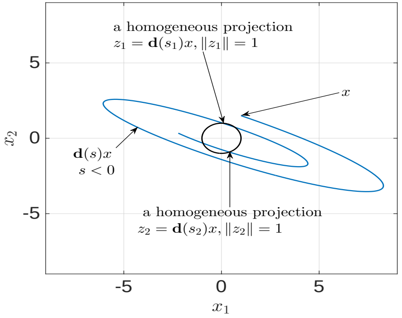
Monotonicity of linear dilations



Monotonicity of linear dilations

Definition

A dilation $\mathbf{d}$ is **monotone** w.r.t. $\|\cdot\|$ if $s \mapsto \|\mathbf{d}(s)x\|$ is strictly increasing, $\forall x \neq \mathbf{0}$.



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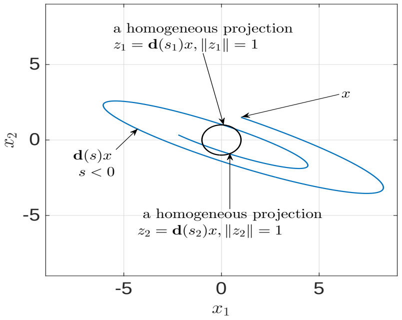
Theorem (Monotonicity in $\mathbb{R}^n$)

A dilation $\mathbf{d}(s)$ is monotone for $\|x\| = \sqrt{x^\top P x}$ **if and only if** $PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, P \succ 0.$



Monotonicity of linear dilations

Definition

A dilation $\mathbf{d}$ is **monotone** w.r.t. $\|\cdot\|$ if $s \mapsto \|\mathbf{d}(s)x\|$ is strictly increasing, $\forall x \neq \mathbf{0}$.

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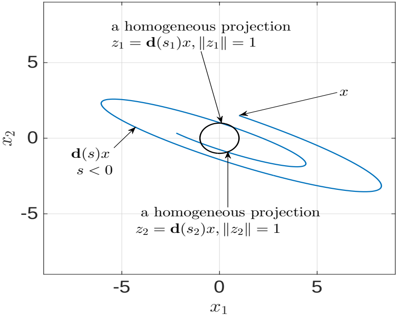


Proposition (Uniqueness of a homogeneous projection to the sphere)

If $\mathbf{d}$ is monotone then $\forall x \neq \mathbf{0}$ there exists a unique pair $(s_0, x_0) \in \mathbb{R} \times \mathcal{S}$ such that $x = \mathbf{d}(s_0)x_0$, where $\mathcal{S} = \{x : \|x\| = 1\}$ is the unit sphere.

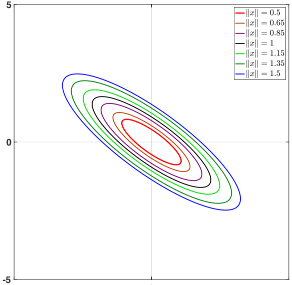
Remark: If $\mathbf{d}(s) = e^s$ then $\frac{x}{\|x\|}$ is standard homogeneous projection.

Homogeneous Norm

Definition (a norm)

$p \in C(\mathbb{R}^n, \mathbb{R}_+)$ is a norm if

- $p(x)=0 \Leftrightarrow x=\mathbf{0}$
- $p(\pm e^s x) = e^s p(x)$
- $p(x+y) \leq p(x) + p(y)$



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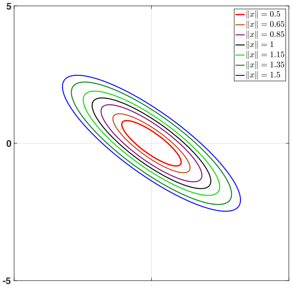

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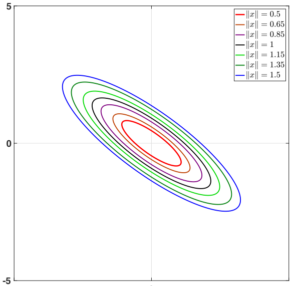
Canonical homogeneous norm for monotone dilation [Polyakov, Coron, Rosier 2016]

$$\|x\|_{\mathbf{d}} = e^{s_x} \quad \text{where} \quad s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0}$$



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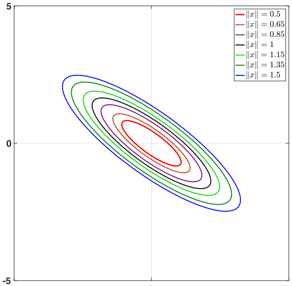
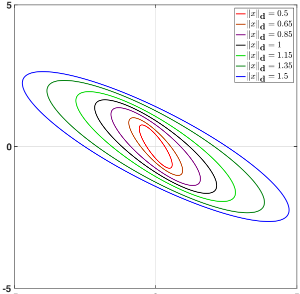




Properties of a canonical homogeneous norm

Let $\|x\|_{\mathbf{d}}$ be induced by $\|x\| = \sqrt{x^\top P x}$ with

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0$$

1) $\|\cdot\|_{\mathbf{d}}$ is positive definite and **d**-homogeneous of degree 1;

2) $$\underline{\sigma}(\|x\|_{\mathbf{d}}) \le \|x\| \le \overline{\sigma}(\|x\|_{\mathbf{d}}),\ \forall x \in \mathbb{R}^n$$

$$\underline{\sigma}(\rho) = \begin{cases} \rho^\beta & \text{if} \quad \rho \le 1, \\ \rho^\alpha & \text{if} \quad \rho > 1, \end{cases} \qquad \overline{\sigma}(\rho) = \begin{cases} \rho^\alpha & \text{if} \quad \rho \le 1, \\ \rho^\beta & \text{if} \quad \rho > 1; \end{cases}$$

$$\alpha = 0.5\lambda_{\max}\left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right) > 0,$$

$$\beta = 0.5\lambda_{\min}\left(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right) > 0.$$

3) $\|\cdot\|_{\mathbf{d}} \in C(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \backslash \{\mathbf{0}\})$ and

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) X^{-1} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) X^{-1} G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}$$

Generalized Homogeneous Vector Space $\mathbb{R}^n_{\mathbf{d}}$

Vector Space

A vector space is a set $\mathbb{V}$ together with two operations (satisfying some axioms):

- a vector addition $\mathbb{V} \times \mathbb{V} \mapsto \mathbb{V}$ denoted by $v \tilde{+} w \in \mathbb{V}$ for $v, w \in \mathbb{V}$.
- a multiplication by a scalar $\mathbb{R} \times \mathbb{V} \mapsto \mathbb{V}$ denoted by $\alpha \tilde{\cdot} v$ for $\alpha \in \mathbb{R}$ and $v \in \mathbb{V}$.

Homeomorphism on $\mathbb{R}^n$

$$\Phi(x) = \|x\|_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad x \in \mathbb{R}^n \qquad \Phi^{-1}(z) = \mathbf{d}(\ln \|z\|)\frac{z}{\|z\|}, \quad z \in \mathbb{R}^n$$

Theorem [*Polyakov 2020*]

Let a linear dilation $\mathbf{d}$ in $\mathbb{R}^n$ be monotone with respect to a norm $\|\cdot\|$ and

- $x \tilde{+} y := \Phi^{-1}(\Phi(x) + \Phi(y)), \quad$ where $x, y \in \mathbb{R}^n$,
- $\lambda \tilde{\cdot} x := \text{sign}(\lambda)\mathbf{d}(\ln |\lambda|)x, \qquad$ where $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$,

Then the set $\mathbb{R}^n$ with the operations $\tilde{+}$ and $\tilde{\cdot}$ is a vector space $\mathbb{R}^n_{\mathbf{d}}$ with the norm $\|\cdot\|_{\mathbf{d}}$.

How to compute $\|x\|_{\mathbf{d}}$?

Bisection Method:

Algorithm $(\|x\|_{\mathbf{d}} = r : \|\mathbf{d}(-\ln r)x\| = 1)$

Initialization: $\underline{r} > 0$ and $\overline{r} > 0$

Step:

if $\|\mathbf{d}(-\ln \overline{r})x\| > 1$ **then**

$\underline{r} = \overline{r}$; $\overline{r} = \min(2\overline{r}, r_{\max})$

else if $\|\mathbf{d}(-\ln \underline{r})x\| < 1$ **then**

$\overline{r} = \underline{r}$; $\underline{r} = \max(\underline{r}/2, r_{\min})$

else

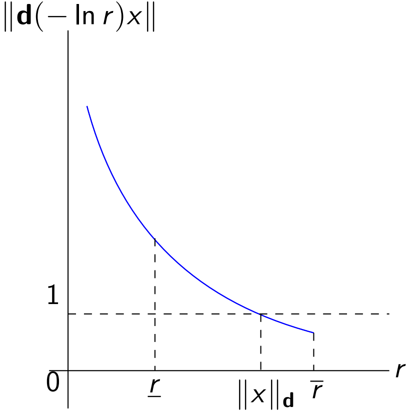
for $j = 1, ..., j_{\max}$ **do**

$c = (\underline{r} + \overline{r})/2$

if $\|\mathbf{d}(-\ln c)x\| < 1$ **then** $\overline{r} = c$

else $\underline{r} = c$

return $\underline{r}, \overline{r}$



after several steps $\|x\|_{\mathbf{d}} \approx (\underline{r} + \overline{r})/2$ if $r_{\min} \leq \|x\|_{\mathbf{d}} \leq r_{\max}$

4. Generalized Homogeneous Sliding Mode Control

Problem Statement

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

- $x(t) \in \mathbb{R}^n$ is the system state,
- $u(t) \in \mathbb{R}^m$ is the control input
- $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ is a known and controllable pair of matrices

Problem 1 is to design a sliding mode control $\tilde{u} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that the system

$$\dot{x}(t) = Ax(t) + B\tilde{u}(x(t)) \tag{2}$$

is globally asymptotically stable and $x = \mathbf{0}$ is the only sliding (discontinuity) set[2].

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- $g : \mathbb{R} \times \mathbb{R}^m$ is an unknown function

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Problem 2 is to characterize a class of uncertainties g to be rejected by $\tilde{u}$.

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A "homogeneous intuition" behind the sliding mode control

- The unit sliding mode control (*Gutman & Leitmann 1976, Utkin 1992*) is given by:

$$\tilde{u} = \tilde{K}_0 x + \tilde{K}\frac{Cx}{\|Cx\|}, \qquad \tilde{K}_0 = -(CB)^{-1}CA, \quad \tilde{K} \in \mathbb{R}^{m\times m}$$

where $C \in R^{m\times n}$: $\det(CB) \neq 0$.

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$$\dot{\sigma} = C\dot{x} = \underbrace{(CA + CB\tilde{K}_0)}_{=0} x - CB\tilde{K}\frac{\sigma}{\|\sigma\|} \qquad \text{is standard homogeneous system!}$$

Notice that $\frac{\sigma}{\|\sigma\|}$ is the standard homogeneous projection to the unit sphere.

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- If $\sigma = x$ then the generalized homogeneous sliding mode control [*Polyakov 2020*]

$$\tilde{u} = K_0 x + K\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x, \quad K_0, K \in \mathbb{R}^{m\times n}$$

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Question 1: How to select K_0 and a dilation $\mathbf{d}$ such that the system

$$\dot{x} = (A + BK_0)x + BK\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x \qquad \text{will be } \mathbf{d}\text{-homogeneous?}$$

Question 2: How to select K to guarantee finite-time stability of the latter system?

Main Theorem

If a pair $\{A, B\}$ is controllable then

1) the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0}, \quad Y_0 \in \mathbb{R}^{m\times n}, \quad G_0 \in \mathbb{R}^{n\times n} \tag{3}$$

always has solutions and for any solution one holds

- $G_{\mathbf{d}} = I_n - G_0$ is anti-Hurwitz and $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ is a linear dilation in $\mathbb{R}^n$,
- the matrix $A_0 = A + BK_0$ is nilpotent for $K_0 = Y_0(I_n - G_0)^{-1}$ and $A_0\mathbf{d}(s) = e^{-s}\mathbf{d}(s)A_0$;

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2) the following LMI has a solution $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$::

$$A_0X + XA_0^\top + BY + Y^\top B^\top + G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top = \mathbf{0}, \quad G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X = X^\top \succ 0; \tag{4}$$

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4) $x = 0$ is the only sliding (discontinuity) set and $u \in C^\infty(\mathbb{R}^n \backslash \{\mathbf{0}\}, \mathbb{R}^m)$ is bounded:

$$\|u\|_{\mathbb{R}^m} \leq \|K_0\| \cdot \|x\| + \|K\|$$

A sketch of the proof

- If $G_{\mathbf{d}} = I_n - G_0, \mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ then $A_0 = A + BK_0$ is nilpotent and

$$A_0\mathbf{d}(s) = e^{-s}\mathbf{d}(s)A_0, \qquad \mathbf{d}(s)B = e^{s}B$$

- $\|x\|_{\mathbf{d}}$ is a Lyapunov and the settling-time function, simultaneously:

$$\frac{d}{dt}\|x\|_{\mathbf{d}} = \overbrace{\|x\|_{\mathbf{d}}\frac{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})}{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}}^{\frac{\partial\|x\|_{\mathbf{d}}}{\partial x}}\overbrace{(A_0 + BYX^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}}))x}^{\dot{x}}$$

$$= \frac{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}(\overbrace{A_0X + XA_0^\top + BY + Y^\top B^\top}^{-(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top)})X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x}{x^\top\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}(\underbrace{G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top}_{\succ 0})X^{-1}\mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})x} = -1.$$

Characterization of uncertainties

Corollary

Let $u : \mathbb{R}^n \mapsto \mathbb{R}^m$ be defined as in the Main Theorem and a locally measurable locally bounded function $g : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfy the following inequality

$$\sup_{t\geq 0, x\in\mathbb{R}^n} \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})g(t,x)}{x^\top \mathbf{d}^\top(-\ln\|x\|_{\mathbf{d}})X^{-1}G_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x} = \kappa < 1, \tag{6}$$

then the origin of the system

$$\dot{x} = Ax + Bu(x) + g(t,x), \quad t > 0 \tag{7}$$

is globally uniformly finite-time stable and

$$\frac{d\|x\|_{\mathbf{d}}}{dt} \leq -1 + \kappa, \quad x \neq \mathbf{0}$$

Remark on rejection of bounded matched perturbation

For $g(t,x) = Bg_0(t,x)$ (6) holds if $g_0^\top B^\top X^{-1} B g_0 < \frac{1}{4}\lambda^2_{\min}\left(X^{-1/2}G_{\mathbf{d}}X^{1/2} + X^{1/2}G_{\mathbf{d}}^\top X^{-1/2}\right)$

"Upgrading" Liner Feedback to Homogeneous SMC

$$\dot{x} = Ax + Bu, \qquad u = Kx \quad \text{is already given}$$

Note: $\alpha = \beta = 1 \quad \Rightarrow \quad u = Kx \qquad\qquad \alpha = 0, \beta = +\infty \quad \Rightarrow \quad u = K_0 x + (K - K_0)\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$

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Algorithm

1. Find the matrices $K_0 \in \mathbb{R}^{m\times n}$ and $G_{\mathbf{d}}$ as in the Main Theorem.
2. Find a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{cases} (A+BK)^\top P + P(A+BK) \prec 0, \\ PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0 \end{cases}$$

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3. Define the control as follows

$$u = K_0 x + (K - K_0)\mathbf{d}(-\ln \mathrm{sat}_{\alpha,\beta}\|x\|_{\mathbf{d}})x$$

where $\|x\|_{\mathbf{d}}$ is induced by $\|x\| = \sqrt{x^\top P x}$ and $\mathrm{sat}_{\alpha,\beta}(\rho) = \begin{cases} \alpha & \text{if} \quad 0 < \rho < \alpha, \\ \rho & \text{if} \quad \alpha \le \rho \le \beta, \\ \beta & \text{if} \quad \rho > \beta. \end{cases}$

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2. Find a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{cases} (A+BK)^\top P + P(A+BK) \prec 0, \\ PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0 \end{cases}$$

3. Define the control as follows

$$u = K_0 x + (K - K_0) \qquad\qquad x$$

"Upgrading" Liner Feedback to Homogeneous SMC

$$\dot{x} = Ax + Bu, \qquad u = Kx \quad \text{is already given}$$

Algorithm

1. Find the matrices $K_0 \in \mathbb{R}^{m\times n}$ and $G_{\mathbf{d}}$ as in the Main Theorem.
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$$\begin{cases} (A+BK)^\top P + P(A+BK) \prec 0, \\ PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0 \end{cases}$$

3. Define the control as follows

$$u = K_0 x + (K - K_0)\mathbf{d}(-\ln \mathrm{sat}_{\alpha,\beta}\|x\|_{\mathbf{d}})x$$

where $\|x\|_{\mathbf{d}}$ is induced by $\|x\| = \sqrt{x^\top P x}$ and $\mathrm{sat}_{\alpha,\beta}(\rho) = \begin{cases} \alpha & \text{if} \quad 0 < \rho < \alpha, \\ \rho & \text{if} \quad \alpha \leq \rho \leq \beta, \\ \beta & \text{if} \quad \rho > \beta. \end{cases}$

Note: $\alpha=\beta=1 \quad \Rightarrow \quad u=Kx \qquad \alpha=0, \beta=+\infty \quad \Rightarrow \quad u=K_0x+(K-K_0)\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$

5. Control Experiment

List of control experiments

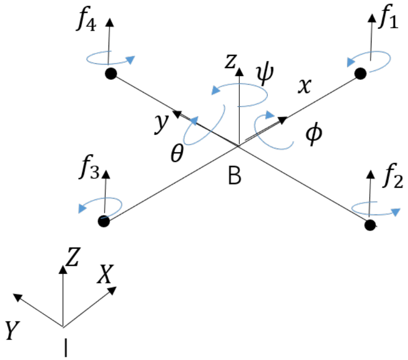
- Rotary Inverted Pendulum System
  Cruz-Ortiz, Ballesteros, Polyakov, Efimov, Chairez, Poznyak, IEEE TIE, 2021
  Institutions: Inria, France+CINVESTAV, Mexico

- Quadrotor Control
  Wang, Polyakov, Zheng, ICRA 2020
  Institution: *Inria, France*

- Two Rotor System
  Zimenko, Polyakov, Efimov, Perruquetti, IEEE TAC, 2020
  Institutions: *Inria, France + ITMO University, Russia*

- Omni-Directional Mobile Robot
  Zhou, Rios, Mera, Zheng, Polyakov, in preparation
  Institution: *Inria, France + TechLaguna, Mexco*

Homogeneous Quadrotor Control (QDrone of Qunaser^{TM})



- f_1, f_2, f_3, f_4 are propellers thrusts (control inputs)
- θ, ϕ, ψ are yaw, pitch and roll angles
- (x, y, z) is position of quadrotor

Results of "upgrade" are shown in http://researchers.lille.inria.fr/~polyakov/drone.mp4

Summary

- The homogeneity is a dilation symmetry known since 18th century.

- Potential advantages of homogeneous control systems vs linear:
    - faster convergence
    - better robustness
    - smaller overshoot (no "peaking" effect)

- The generalized homogeneity is useful for analysis of high-order sliding mode control systems [*Levant 2003, Orlov 2005, Moreno 2010,...*]

- The generalized homogeneity can be utilized for sliding mode control design as well!

Thank you very much for your attention